UNIT - I

INTERPOLATION

→ Interpolation is the process of finding the most appropriate estimate for missing data for making most probable estimate it requires.

① The frequency distribution is normal and not marked by ups and downs.

② Interpolation is a technique is used in various disciplines like economics, business, population, studies etc. It's used to fill the gaps in the satisfied data for the sake of continuity of info.

Given tabular form.

$x: x_0 \quad x_1 \quad x_2 \quad --- \quad x_n$

$y: y_0 \quad y_1 \quad y_2 \quad --- \quad y_n$

Satisfy the relation $y = f(x)$ and the explicit function of $f(x)$ is normal then $f(x)$ can be replaced by a simpler function $\phi(x)$ such that $f(x)$ & $\phi(x)$ agree with the set of tabulated point and any other value may be calculated from $\phi(x)$. The process of finding $\phi(x)$ is known as interpolation and $\phi(x)$ is known as interpolation function.

Here we find various interpolation polynomial using the concepts of forward difference, backward difference and central difference

Forward difference:-

Consider a function $y = f(x)$ of an independent variable x. Let $y_0, y_1, y_2 \cdots y_r$ be the values of y corresponding to values $x_0, x_1, x_2 \cdots x_r$ of x resp. then the difference $y_1-y_0, y_2-y_1, y_3-y_2 \cdots y_{r+1}-y_r$ are called the first forward differences of y and we denote them by $\Delta y_0, \Delta y_1, \Delta y_2 \cdots \Delta y_r$ and Here, the symbol Δ is called forward difference operator.

$$\Delta y_0 = y_1 - y_0 \,;\ \Delta y_1 = y_2 - y_1 \ \cdots\ \Delta y_r = y_{r+1} - y_r.$$

where $r = 0, 1, 2 \cdots$

The difference of 1st order differences are called second order differences.

$$\Delta^2 y_0 = \Delta y_1 - \Delta y_0 \,,\ \Delta^2 y_1 = \Delta y_2 - \Delta y_1 \cdots$$

$$\Delta^2 y_r = \Delta y_{r+1} - \Delta y_r \quad \text{where } r = 0, 1, 2 \cdots$$

The n^{th} forward differences and defined by formula.

$$\Delta^n y_r = \Delta^{n-1} y_{r+1} - \Delta^{n-1} y_r \quad \text{where } r = 0, 1, 2 \cdots$$

Note:- $\Delta f(x) = f(x+h) - f(x)$; $h = 1, 2, 3 \cdots$

<u>forward difference table</u>:-

x y.

x_0 y_0

$\quad\quad y_1 - y_0 = \Delta y_0$

x_1 y_1 $\Delta y_1 - \Delta y_0 = \Delta^2 y_0$

$\quad\quad y_2 - y_1 = \Delta y_1$ $\Delta^2 y_1 - \Delta^2 y_0 = \Delta^3 y_0$

x_2 y_2 $\Delta y_2 - \Delta y_1 = \Delta^2 y_1$

$\quad\quad y_3 - y_2 = \Delta y_2$

x_3 y_3

Backward differences:-

Let $y_0, y_1, y_2 \cdots y_r$ be the values of function $y = f(x)$ corresponding to values of $x_0, x_1, x_2 \cdots x_r$ of x respectively thus

$\nabla y_1 = y_1 - y_0$, $\nabla y_2 = y_2 - y_1$ ---- are called the first backward differences.

In general $\nabla y_r = y_r - y_{r-1}$ $r = 1, 2 \cdots$

The symbol ∇ is called backward differences operator.

The difference of 1st order backward differences are called second order backward differences.

$$\nabla^2 y_2 = \nabla y_2 - \nabla y_1, \quad \nabla^2 y_3 = \nabla y_3 - \nabla y_2 \cdots$$

In general $\nabla^2 y_r = \nabla y_r - \nabla y_{r-1}$ $r = 1, 2, 3 \cdots$

The n^{th} backward difference are defined by

$$\nabla^n y_r = \nabla^{n-1} y_r - \nabla^{n-1} y_{r-1}, \quad r = n, n+1 \cdots$$
$$n = 1, 2, 3 \cdots$$

Note: $\nabla f(x) = f(x) - f(x-h)$

Backward difference table:-

x	y			
x_0	y_0			
		$\nabla y_1 = y_1 - y_0$		
x_1	y_1		$\nabla^2 y_2 = \nabla y_2 - \nabla y_1$	
		$\nabla y_2 = y_2 - y_1$		$\nabla^3 y_3 = \nabla^2 y_3 - \nabla^2 y_2$
x_2	y_2		$\nabla^2 y_3 = \nabla y_3 - \nabla y_2$	
		$\nabla y_3 = y_3 - y_2$		
x_3	y_3			

Central differences :-

Given $x: x_0 \quad x_1 \quad x_2 \quad x_3 \quad \cdots \quad x_r$

$y: y_0 \quad y_1 \quad y_2 \quad y_3 \quad \cdots \quad y_r$

We define the 1st central differences as follows.

$\delta y_{1/2}, \delta y_{3/2}, \delta y_{5/2} \cdots$ as follows.

$\delta y_{1/2} = y_1 - y_0, \quad \delta y_{3/2} = y_2 - y_1 \quad \cdots \quad \delta y_{r-1/2} = y_r - y_{r-1}$

The symbol δ is called central differences operator.

The second central difference are the difference of 1st order central differences.

$\delta^2 y_1 = \delta y_{3/2} - \delta y_{1/2} \quad \cdots \quad \delta^2 y_r = \delta y_{r+\frac{1}{2}} - \delta y_{r-1/2}$

In general, n^{th} central differences are given by

① for odd n: $\delta^n y_{r-1/2} = \delta^{n-1} y_r - \delta^{n-1} y_{r-1}$

② for even n: $\delta^n y_r = \delta^{n-1} y_{r+\frac{1}{2}} - \delta^{n-1} y_{r-\frac{1}{2}}$

Note :- $\delta f(x) = f(x + \frac{1}{2}h) - f(x - \frac{1}{2}h)$

Central difference table.

$x \quad y$

$x_0 \quad y_0$

$\qquad \delta y_{1/2} = y_1 - y_0$

$x_1 \quad y_1 \qquad \delta^2 y_1 = \delta y_{3/2} - \delta y_{1/2}$

$\qquad \delta y_{3/2} = y_2 - y_1 \qquad \delta^3 y_{3/2} = \delta^2 y_2 - \delta^2 y_1$

$x_2 \quad y_2 \qquad \delta^2 y_2 = \delta y_{5/2} - \delta y_{3/2}$

$\qquad \delta y_{5/2} = y_3 - y_2$

$x_3 \quad y_3$

Average operator :- The average operator 'μ' is defined by

$$\mu y_r = \frac{1}{2}\left[y_{r+\frac{1}{2}} + y_{r-\frac{1}{2}}\right]$$

Note :- $\mu f(x) = \frac{1}{2}\left[f(x+\frac{h}{2}) + f(x-\frac{h}{2})\right]$.

Ex :- $\mu y_1 = \frac{1}{2}\left[y_{3/2} + y_{1/2}\right]$.

Shift operator :- Shift operator is defined by

$$E y_r = y_{r+1}$$

$$E^2 y_r = y_{r+2}$$

$$E^n y_r = y_{r+n}$$

Inverse shift operator E^{-1} is defined by

$$E^{-n} y_r = y_{r-n}.$$

Note :- $E f(x) = f(x+h)$.

Relations of symbols.

① Prove that $\Delta = E - 1$

Proof :- we have $\Delta y_0 = y_1 - y_0$.

$$\Delta y_0 = E y_0 - y_0$$

$$\Delta y_0 = (E-1) y_0$$

$$\boxed{\Delta = E - 1}$$

$$\boxed{E = 1 + \Delta}$$

Ex :- $\Delta^2 y_0 = (E-1)^2 y_0.$

$$= (E^2 + 1 - 2E) y_0.$$

$$= E^2 y_0 + y_0 - 2E y_0.$$

$$= y_2 + y_0 - 2y_1.$$

(2) P.T $\nabla = 1 - E^{-1}.$

proof :- we have $\nabla y_1 = y_1 - y_0.$

$$\nabla y_1 = y_1 - E^{-1} \cdot y_1$$

$$\nabla y_1 = y_1 (1 - E^{-1})$$

$$\boxed{\nabla = 1 - E^{-1}}$$

$$\boxed{E^{-1} = 1 - \nabla}$$

(3) P.T $\delta = E^{1/2} - E^{-1/2}.$

we have $\delta y_{1/2} = y_1 - y_0.$

$$\delta y_{1/2} = E^{1/2} y_{1/2} - E^{-1/2} y_{1/2}.$$

$$\delta \cancel{y_{1/2}} = \cancel{y_{1/2}} [E^{1/2} - E^{-1/2}]$$

$$\boxed{\delta = E^{1/2} - E^{-1/2}}$$

(4) S.T $\mu = \frac{1}{2}(E^{1/2} + E^{-1/2}].$

$$\mu y_r = \frac{1}{2}[y_{r+\frac{1}{2}} + y_{r-\frac{1}{2}}]$$

$$\mu y_r = \frac{1}{2}(E^{1/2} y_r + E^{-1/2} y_r]$$

$$\mu \cancel{y_r} = \cancel{y_r} [\tfrac{1}{2}[E^{1/2} + E^{-1/2}]]$$

$$\boxed{\mu = \frac{1}{2}[E^{1/2} + E^{-1/2}]}$$

(5) P.T $\mu^2 = 1 + \frac{1}{4}\delta^2$.

We have $\mu = \frac{1}{2}[E^{1/2} + E^{-1/2}]$

$$\mu^2 = \left[\frac{1}{2}[E^{1/2} + E^{-1/2}]\right]^2$$

$$= \frac{1}{4}[E^{1/2} + E^{-1/2}]^2$$

$$= \frac{1}{4}\left[[E^{1/2} - E^{-1/2}]^2 + 4\right]$$

$$= \frac{1}{4}[E^{1/2} - E^{-1/2}]^2 + 1$$

$$\boxed{\mu^2 = 1 + \frac{1}{4}\delta^2}$$

<u>operator D</u> :-

The operator D is defined by

$$D f(x) = \frac{d}{dx} f(x).$$

We know that $y(x+h) = y(x) + h y'(x) + \frac{h^2}{2!} y''(x) + \cdots$

$$= y(x) + h D(y(x)) + \frac{h^2}{2!} D^2 y(x) + \cdots$$

$$E y(x) = \left[1 + hD + \frac{h^2 D^2}{2!} + \frac{h^3 D^3}{3!} + \cdots\right] y(x)$$

$$E = 1 + hD + \frac{(hD)^2}{2!} + \frac{(hD)^3}{3!} - \cdots$$

$$\boxed{E = e^{hD}}$$

$$\boxed{1 + \Delta = e^{hD}}$$

$$\boxed{\log(1+\Delta) = hD}$$

Problems :-

① Proof :- $E\nabla = \Delta = \nabla E$.

$$E\nabla(f(x)) = E[\nabla f(x)]$$
$$= E[f(x) - f(x-h)]$$
$$= Ef(x) - E(f(x-h))$$
$$= f(x+h) - f(x-h+h)$$
$$= f(x+h) - f(x)$$
$$= \Delta f(x)$$

$\Delta = \nabla E$

$$\nabla E y_1 = (\nabla y_1)E$$
$$= (y_1 - y_0)E$$
$$= Ey_1 - Ey_0$$
$$= y_2 - y_1$$
$$\nabla E y_1 = \Delta y_1$$
$$\nabla E = \Delta$$

$$E\nabla y_1 = E[\nabla y_1]$$
$$= E[y_1 - y_0]$$
$$= Ey_1 - Ey_0$$
$$= y_2 - y_1$$
$$= \Delta y_1 \Rightarrow \boxed{E\nabla = \Delta}$$

② P.T $\delta E^{1/2} = \Delta$.

we have $\delta = E^{1/2} - E^{-1/2}$.

$$\delta E^{1/2} = (E^{1/2} - E^{-1/2})E^{1/2}$$
$$= E - 1$$
$$= \Delta$$

③ PT $1 + \mu^2\delta^2 = (1 + \frac{1}{2}\delta^2)^2$.

$$LHS = 1 + \mu^2\delta^2$$
$$= 1 + \left(\frac{1}{2}(E^{1/2} + E^{-1/2})^2\right)(E^{1/2} - E^{-1/2})^2$$
$$= 1 + \frac{1}{4}\left[(E^{1/2})^2 - (E^{-1/2})^2\right]^2$$

$$= 1 + \frac{1}{4}\left[E - E^{-1}\right]^2.$$

$$= \frac{4 + [E - E^{-1}]^2}{4}.$$

$$= \frac{(E + E^{-1})^2}{4}.$$

$$RHS = \left(1 + \frac{1}{2}\delta^2\right)^2.$$

$$= \left[1 + \frac{1}{2}\left(E^{1/2} - E^{-1/2}\right)^2\right]^2.$$

$$= \left[1 + \frac{1}{2}\left(E + E^{-1} - 2\right)\right]^2$$

$$= \left[\frac{\cancel{2} + E + E^{-1} - \cancel{2}}{2}\right]^2$$

$$= \frac{(E + E^{-1})^2}{4}.$$

④ : P.T $\mu\delta = \frac{1}{2}\Delta E^{-1} + \frac{1}{2}\Delta.$

$$RHS = \frac{1}{2}\Delta\left(E^{-1} + 1\right)$$

$$= \frac{1}{2}[E - 1][E^{-1} + 1]$$

$$= \frac{1}{2}[\cancel{1} + E - E^{-1} - \cancel{1}]$$

$$= \frac{1}{2}[E - E^{-1}] \longrightarrow ①$$

$$LHS = \mu\delta = \frac{1}{2}\left(E^{1/2} + E^{-1/2}\right)\left[E^{1/2} - E^{-1/2}\right]$$

$$= \frac{1}{2}[E - E^{-1}] \; - ②$$

LHS = RHS.

Hence proved

⑤ P.T $\Delta = \frac{1}{2}\delta^2 + \delta\sqrt{1+\frac{\delta^2}{4}}$.

sol $\frac{1}{2}\delta^2 + \delta\sqrt{1+\frac{\delta^2}{4}}$.

$= \frac{1}{2}\delta^2 + \frac{1}{2}\delta\sqrt{4+\delta^2}$.

$= \frac{\delta}{2}\left[\delta + \sqrt{4+\delta^2}\right]$

$= \frac{\delta}{2}\left[E^{1/2} - E^{-1/2} + \sqrt{4+(E^{1/2}-E^{-1/2})^2}\right]$

$= \frac{\delta}{2}\left[E^{1/2} - E^{-1/2} + \sqrt{(E^{1/2}+E^{-1/2})^2}\right]$

$= \frac{\delta}{2}\left[E^{1/2} - \cancel{E^{-1/2}} + E^{1/2} + \cancel{E^{-1/2}}\right]$

$= \frac{\delta}{\cancel{2}}\left[\cancel{2}\cdot E^{1/2}\right]$

$= (E^{1/2} - E^{-1/2})E^{1/2}$.

$= E - 1 = \Delta$.

⑥ The following table gives a set of values of x and corresponding values of $y = f(x)$

x	10	15	20	25	30	35
y	19.97	21.51	22.47	23.52	24.65	25.89.

Form the forward difference table and write down the values of $\Delta f(10)$, $\Delta^2 f(10)$, $\Delta^3 f(15)$, $\Delta^4 f(15)$.

Sol.

x	f(x)					
10	19.97	1.54	−0.58	0.67	−0.68	0.72
15	21.51	0.96	0.09	−0.01	0.04	
20	22.47	1.05	0.08	0.03		
25	23.52	1.13	0.11			
30	24.65	1.24				
35	25.89					

Q. construct a forward difference table from the following data:

x	0	1	2	3	4
y	1	1.5	2.2	3.1	4.6

Evaluate $\Delta^3 y_1$.

x	f(x)	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
x_0 0	y_0 1	Δy_0 0.5	$\Delta^2 y_0$ 0.2	$\Delta^3 y_0$ 0	$\Delta^4 y_0$ 0.4
x_1 1	y_1 1.5	Δy_1 0.7	$\Delta^2 y_1$ 0.2	$\Delta^3 y_1$ 0.4	
x_2 2	y_2 2.2	Δy_2 0.9	$\Delta^2 y_2$ 0.6		
x_3 3	y_3 3.1	Δy_3 1.5			
x_4 4	y_4 4.6				

$\Delta^3 y_1 = 0.4$

⑧ Find the missing term in the following data.

x	0	1	2	3	4
y	1	3	9	-	81

Sol: Here 4 values are known.

$\Rightarrow$ Third differences are constant.

$\Rightarrow$ fourth differences are zero.

consider $\Delta^4 y_0 = 0$.

$$(E-1)^4 y_0 = 0$$

$$[{}^4C_0 E^4 - {}^4C_1 E^3 + {}^4C_2 E^2 - {}^4C_3 E + {}^4C_4 E^0] y_0 = 0$$

$$\Rightarrow [E^4 - 4E^3 + 6E^2 - 4E + 1] y_0 = 0$$

$$E^4 y_0 - 4E^3 y_0 + 6 y_0 E^2 - 4 y_0 E + y_0 = 0$$

$$y_4 - 4y_3 + 6 y_2 - 4y_1 + y_0 = 0$$

$$81 - 4y_3 + 6(9) - 4(3) + 1 = 0$$

$$\boxed{y_3 = 31}$$

⑨ Find the missing term from following data.

x	1	2	3	4	5
y	2	5	7	-	32

Here 4 values are known.

$\Rightarrow$ Third differences are constant.

$\Rightarrow$ fourth differences are zero.

$$\Delta^4 y_0 = 0.$$

$$(E-1)^4 y_0 = 0.$$

$$\left[{}^4C_0 E^4 - {}^4C_1 E^3 + {}^4C_2 E^2 - {}^4C_3 E + {}^4C_4 E^0\right] y_0 = 0.$$

$$\left[E^4 - 4E^3 + 6E^2 - 4E + 1\right] y_0 = 0.$$

$$E^4 y_0 - 4E^3 y_0 + 6E^2 y_0 - 4E y_0 + y_0 = 0.$$

$$y_4 - 4y_3 + 6y_2 - 4y_1 + y_0 = 0.$$

$$32 - 4y_3 + 6(7) - 4(5) + 2 = 0.$$

$$32 - 4y_3 + 42 - 20 + 2 = 0.$$

$$\cancel{4}y_3 = \cancel{56}\ 14.$$

$$\boxed{y_3 = 14}$$

(10) Find the missing terms in table in following table.

x	1	2	3	4	5	6	7	8
y	1	8	—	64	—	216	343	512
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7

$$\Delta^6 y_0 = 0 \ \& \ \Delta^6 y_1 = 0.$$

$$\Delta^6 y_0 = 0.$$

$$(E-1)^6 y_0 = 0.$$

$$\left({}^6C_0 E^6 - {}^6C_1 E^5 + {}^6C_2 E^4 - {}^6C_3 E^3 + {}^6C_4 E^2 - {}^6C_5 E + {}^6C_6 E^0\right) y_0 = 0$$

$$\left(E^6 - 6E^5 + 15E^4 - 20E^3 + 15E^2 - 6E + 1\right) y_0 = 0.$$

$$y_6 - 6y_5 + 15y_4 - 20y_3 + 15y_2 - 6y_1 + y_0 = 0. \quad 152$$

$$343 - 6(216) + 15y_4 - 20(64) + 15y_2 - 6(8) + 1 = 0 \Rightarrow 15(y_4 + y_2) = 228$$

$\boxed{y_4 + y_2 = 152}$ —— ①

$\Delta^6 y_1 = 0$

$E^n y_r = y_{r+n}$

$(E-1)^6 y_1 = 0$

$\Rightarrow [E^6 - 6E^5 + 15E^4 - 20E^3 + 15E^2 - 6E + 1] y_1 = 0$

$y_7 - 6y_6 + 15y_5 - 20y_4 + 15y_3 - 6y_2 + y_1 = 0$

$512 - 6(343) + 15(216) - 20y_4 + 15(64) - 6y_2 + 8 = 0$

$-20y_4 - 6y_2 = -2662$

$20y_4 + 6y_2 = 2662$ —— ②

$y_4 + y_2 = 152$ —— ①

solve ① & ②

$\boxed{y_4 = 125}$

$\boxed{y_2 = 27}$

(11) If $f(x) = x^3 + 5x - 7$ form a table of forward difference taking $x = -1, 0, 1, 2, 3, 4, 5$. show that third differences are constant.

x	-1	0	1	2	3	4	5
y	-13	-7	-1	11	35	77	143

x	$f(x)$ y				
-1	-13	6	0		
0	-7	6	6	6	
1	-1	12	12	6	
2	11	24	18	6	
3	35	42	24	6	
4	77	66			
5	143				

Hence 3rd differences are constant.

$\cos C - \cos D$
$= -2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$
$= 2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{D-C}{2}\right)$

(12) Evaluate ① $\Delta \cos x$ ② $\Delta \tan^{-1}x$ ③ $\Delta^n e^{ax+b}$.

① $\Delta \cos x = \cos(x+h) - \cos x$

$= 2\sin\left(\frac{x+h+x}{2}\right)\sin\left(\frac{x+h-\cancel{x}}{2}\right)$

$= 2\sin\left(\frac{2x+h}{2}\right)\sin\left(\frac{h}{2}\right)$

② $\Delta \tan^{-1}x = \tan^{-1}\frac{x}{x+h} - \tan^{-1}\frac{y}{x}$

$= \tan^{-1}\left(\frac{x+h-x}{1+x(x+h)}\right)$

$= \tan^{-1}\left(\frac{h}{1+x(x+h)}\right)$

$\boxed{\tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)}$

③ $\Delta^n e^{ax+b}$

$\Delta e^{ax+b} = e^{a(x+h)+b} - e^{ax+b}$

$= e^{ax+ah+b} - e^{ax+b}$

$= e^{ax+b} \cdot e^{ah} - e^{ax+b}$

$= e^{ax+b}(e^{ah} - 1)$

$$\Delta^2 e^{ax+b} = \Delta(\Delta e^{ax+b})$$

$$= \Delta(e^{ah}-1)(e^{ax+b})$$

$$= e^{ah}-1(\Delta e^{ax+b})$$

$$= (e^{ah}-1)[(e^{ah}-1)(e^{ax+b})].$$

$$(e^{ah}-1)^2 e^{ax+b}.$$

$$\vdots$$

$$\Delta^n e^{ax+b} = (e^{ah}-1)^n e^{ax+b}.$$

(4) If interval of differencing is unity prove.

$$\Delta x(x+1)(x+2)(x+3) = 4(x+1)(x+2)(x+3) \quad \boxed{h=1}$$

$$\Delta x(x+1)(x+2)(x+3) = f(x+1) - f(x)$$

$$= (x+1)(x+2)(x+3)(x+4) - x(x+1)(x+2)(x+3)$$

$$= (x+1)(x+2)(x+3)[\cancel{x}+4-\cancel{x}]$$

$$= 4(x+1)(x+2)(x+3)$$

(14) Find the first difference of the polynomial $x^4-12x^3+42x^2-30x+9$ with interval of differencing is 2.

Given $h=2$.

$$f(x) = x^4-12x^3+42x^2-30x+9.$$

$$\Delta f(x) = f(x+h) - f(x).$$
$$= f(x+2) - f(x).$$

$$\Delta f(x) = (x+2)^4 - 12(x+2)^3 + 42(x+2)^2 - 30(x+2) + 9.$$
$$-(x^4-12x^3+42x^2-30x+9).$$

$$= (x+2)(x^3+8+6x^2+12x - \underbrace{12(x^2+4+4x)}_{-12x^2-48-48x}) + 42x + 84 - 30) + \cancel{9}$$
$$-x^4+12x^3-42x^2+30x \cancel{9}.$$

$$= (x+2)(x^3-6x^2+6x+44) - x^4+12x^3-42x^2+30x.$$
$$= \cancel{x^4} - 6x^3+6x^2+44x+2x^3-12x^2+12x+88 - \cancel{x^4}+12x^3-42x^2+30x.$$
$$= 8x^3-48x^2+88x - 28 //$$

Newton's Forward interpolation (equal interval)

Let $y=f(x)$ be a polynomial of degree n, then Newton's forward interpolation formula is given by.

$$y = f(x) = f(x_0+Ph) = y_0 + P\Delta y_0 + \frac{P(P-1)}{2!}\Delta^2 y_0 + \frac{P(P-1)(P-2)}{3!}\Delta^3 y_0 \cdot$$
$$\cdots \frac{P(P-1)\cdots(P-(n-1))}{n!}\Delta^n y_0.$$

Here $x = x_0 + Ph$. $\quad$ h = interval of difference.

$$P = \frac{x-x_0}{h}.$$

Note :- Newton's forward interpolation is used to interpolate at the begining of the given data.

Newton's backward interpolation.

let $y = f(x)$ be a polynomial of degree n, then Newton's backward interpolation formula is given by

$$y = f(x) = f(x_n + ph) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \dots + \frac{p(p+1)(p+2)\dots(p+(n-1))}{n!}\nabla^n y_n.$$

Note:- Newton's backward interpolation is useful for the end of the tabulated values.

Problems

① The population of a town in the decimal census was given below. Estimate the population for the year 1895, 1925

year x.	1891	1901	1911	1921	1931
population y (Thousands)	46	66	81	93	101

$h = 10 \quad x = 1895 \quad x_0 = 1891.$

$x = x_0 + ph.$

$$p = \frac{x - x_0}{h} = \frac{1895 - 1891}{10} = \frac{4}{10} = 0.4$$

x	y	Δy_0	$\Delta^2 y_0$	$\Delta^3 y_0$	$\Delta^4 y_0$
1891	46 (y_0)				
		20			
1901	66		-5		
		15		2	
1911	81		-3		-3
		12		-1	
1921	93		-4		($\Delta^4 y_4$)
		8	($\Delta^2 y_4$)	($\Delta^3 y_4$)	
1931	101 (y_4)	(Δy_4)			

Newton's forward interpolation.

$$y = f(x) = f(x_0 + ph)$$

$$= y_0 + p\Delta y_0 + \frac{p(p-1)}{2!} \cdot \Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!} \Delta^3 y_0 - - -$$

$$= y_0 + p\Delta y_0 + \frac{p(p-1)}{2!} \Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!} \Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!} \Delta^4 y_0$$

$$= 46 + (0.4)(20) + \frac{(0.4)(0.4-1)}{2}(-5) + \frac{(0.4)(0.4-1)(0.4-2)}{6}(2)$$

$$+ \frac{(0.4)(0.4-1)(0.4-2)(0.4-3)}{24}(-3)$$

$$= 46 + 8 + \frac{(0.4)(-0.6)(-5)}{2} + \frac{(0.4)(-0.6)(-1.6)}{6}(2)$$

$$+ \frac{(0.4)(-0.6)(-1.6)(-2.6)}{24}(-3)$$

$$= \underline{54.8528}.$$

Newton's backward interpolation.

$$y = f(x) = f(x_n + ph)$$

$$= y_n + p\nabla y_n + \frac{p(p+1)}{2!} \nabla^2 y_n + \frac{p(p+1)(p+2)}{3!} \nabla^3 y_n + \frac{p(p+1)(p+2)(p+3)}{4!} \nabla^4 y_n$$

$h = 10$ $\quad x = 1925$ $\quad x_0 = 1931 \Rightarrow x = x_0 + ph.$

$$p = \frac{x - x_0}{h} = \frac{1925 - 1931}{10} = -0.6$$

$$y = 101 + (-0.6)(8) + \frac{(-0.6)(-0.6+1)}{2}(-4) + \frac{(-0.6)(-0.6+1)(-0.6+2)}{6}(-1)$$

$$+ \frac{(-0.6)(-0.6+1)(-0.6+2)(-0.6+3)}{24}(-3)$$

$$y = 101 + (-0.6)(8) + \frac{(-0.6)(0.4)}{2}(-4) + \frac{(-0.6)(0.4)(1.4)}{6}(-1)$$

$$+ \frac{(-0.6)(0.4)(1.4)(2.4)}{24}(-3).$$

$= 96.8368.$

② Applying Newton's forward formula, compute the value of $\sqrt{5.5}$, given that $\sqrt{5} = 2.236$, $\sqrt{6} = 2.449$, $\sqrt{7} = 2.646$, $\sqrt{8} = 2.828$.

x	$f(x) = y = \sqrt{x}$			
5	2.236			
		0.213		
6	2.449		-0.016	
		0.197		0.001
7	2.646		-0.015	
		0.182		
8	2.828			

Newton's law of forward interpolation.

$y = f(x) = f(x_0 + ph)$

$$= y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 \ldots$$

$h = 1$

$x = \cancel{22.230} \cdot 5.5$

$x_0 = 5$

$p = \frac{x - x_0}{h} = \frac{5.5 - 5}{10} = 0.05$

$p = 0.5$

$y = 2.236 + 0.213 + \frac{(0.213)(-0.787)(-0.016)}{2!} +$

$\frac{(0.213)(-0.787)(-1.787)(-0.016)}{6}$

$= 2.3445$

③ Given $\sin 45 = 0.7071$, $\sin 50 = 0.7660$, $\sin 55° = 0.8192$
$\sin 60 = 0.8660$, find $\sin 52°$ using forward

x	45	50	55	60
y	0.7071	0.766	0.8192	0.866

$x = 52$ $x_0 = 45$

$h = \frac{52 - 45}{5} \Rightarrow \frac{7}{5} = 0.14$

x	y			
45	0.7071	0.0589	-5.7×10^{-3}	-0.7×10^{-3}
50	0.766	0.0532	-6.4×10^{-3}	
55	0.8192	0.0468		
60	0.866			

$y = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 \ldots$

$y = 0.7071 + 0.14 \times 0.0589 + \frac{\overset{0.07}{\cancel{0.14}}(-0.86)(-5.7\times10^{-3})}{\cancel{2}}$

$+ \frac{\overset{0.07}{\cancel{0.14}}(-0.86)(-1.86)(-0.7\times10^{-3})}{\cancel{6}\,3}$

③ $y = 707.0\times10^{-5} + 824.6\times10^{-5} + 34.314\times10^{-5} + 2.61268\times10^{-5}$

$y = 71571.5268\times10^{-5}$.

④ 96 $f(1.15) = 1.0723$, $f(1.20) = 1.0954$, $f(1.25) = 1.1180$, $f(1.30) = 1.1401$

find $f(1.28)$.

x	y			
1.15	1.0723			
		0.0231		
1.20	1.0954		-5×10^{-4}	
		0.226		0
1.25	1.1180		-5×10^{-4}	
		0.0221		
1.30	1.1401			

$x = 1.28$ $\quad x_0 = 1.30$.

$$p = \frac{x - x_n}{h} = \frac{-0.02}{5} = -4\times10^{-3}$$

Newtons backward interpolation.

$$y = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{6!}\nabla^3 y_n$$

$$y = 1.1401 + (-4\times10^{-3})(0.0221) + \frac{(-4\times10^{-3})(0.996)}{2}(-5\times10^{-4})$$

$y = 1.1401 - 0.0884\times10^{-3} + 9.96\times10^{-7}$

$y = 1.1400116 + 9.46\times10^{-7}$

$y = 1.140012596$

(5) The following table gives corresponding values of x and y. Construct difference table and then express y as function of x.

x	0	1	2	3	4
y	3	6	11	18	27

Sol

$$\begin{array}{cccccc} x & y & & & & \\ 0 & 3 & & & & \\ & & 3 & & & \\ 1 & 6 & & 2 & & \\ & & 5 & & 0 & \\ 2 & 11 & & 2 & & 0 \\ & & 7 & & 0 & \\ 3 & 18 & & 2 & & \\ & & 9 & & & \\ 4 & 27 & & & & \end{array}$$

$h = 1 \quad x = x \quad x_0 = 0$

$P = \frac{x - x_0}{h} = \frac{x-0}{1} = x$

$P = x$

Newton's forward interpolation

$y = f(x) = y_0 + p\Delta y_0 + \frac{P(P-1)}{2!}\Delta^2 y_0 + \frac{P(P-1)(P-2)}{3!}\Delta^3 y_0 + \cdots$

$y = 3 + x(3) + \frac{x(x-1)}{\cancel{2}}\cancel{(2)}$

$= x^2 - x + 3x + 3$

$y = x^2 + 2x + 3$

(6) Find the newton's forward difference interpolating polynomial for the data.

x	0	1	2	3
y	1	3	7	13

$$\begin{array}{ccccc} x & y & & & \\ 0 & 1 & & & \\ & & 2 & & \\ 1 & 3 & & 2 & \\ & & 4 & & 0 \\ 2 & 7 & & 2 & \\ & & 6 & & \\ 3 & 13 & & & \end{array}$$

$h = 1 \quad x = x \quad x_0 = 0$

$P = \frac{x}{1} = x$

$y = f(x) = y_0 + p\Delta y_0 + \frac{P(P-1)}{2!}\Delta^2 y_0 + \frac{P(P-1)(P-2)}{3!}\Delta^3 y_0$

$\Rightarrow y = 1 + x(2) + \frac{x(x-1)}{2}(2)$

$= 1 + 2x + x^2 - x$

$\boxed{y = x^2 + x + 1}$

Gauss forward interpolation formula is given by.

$$y = f(x) = f(x_0 + ph)$$

$$= y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{p(p-1)(p+1)}{3!}\Delta^3 y_{-1} + \frac{p(p-1)(p+1)(p-2)}{4!}\Delta^4 y_{-2}$$

$$+ \frac{(p+2)(p+1)p(p-1)(p-2)}{5!}\Delta^5 y_{-2}.$$

Gauss backward interpolation is given by.

$$y = f(x) = f(x_0 + ph)$$

$$= y_0 + p\Delta y_{-1} + \frac{p(p+1)}{2!}\Delta^2 y_{-1} + \frac{p(p+1)(p-1)}{3!}\Delta^3 y_{-2} + \frac{p(p+1)(p+2)(p-1)}{4!}\Delta^4 y_{-2}.$$

$$+ \frac{p(p+1)(p+2)(p-1)(p-2)}{5!}\Delta^5 y_{-3} + \cdots$$

Central difference

x	y				
x_{-2}	y_{-2}				
		Δy_{-2}			
x_{-1}	y_{-1}		$\Delta^2 y_{-2}$		
		Δy_{-1}		$\Delta^3 y_{-2}$	
x_0	y_0		$\Delta^2 y_{-1}$		$\Delta^4 y_{-2}$
		Δy_0		$\Delta^3 y_{-1}$	
x_1	y_1		$\Delta^2 y_0$		
		Δy_1			
x_2	y_2				

① Problem: From the following table values of x & $y = e^x$ interpolate values of y when $x = 1.91$.

x	1.7	1.8	1.9	2	2.1	2.2
y	5.4739	6.0496	6.6859	7.3891	8.1662	9.0250

x	y	Δy_{-2}	$\Delta^2 y_{-2}$	$\Delta^3 y_{-2}$	$\Delta^4 y_{-2}$	$\Delta^5 y_{-2}$
1.7 x_{-2}	5.4739 y_{-2}					
		0.5757				
1.8 x_{-1}	6.0496 y_{-1}		0.0606			
		0.6363 Δy_{-1}		0.0063		
1.9 x_0	6.6859 y_0		0.0669 $\Delta^2 y_{-1}$		0.0007	
		0.7032 Δy_0		0.007 $\Delta^3 y_{-1}$		0.0001
2 x_1	7.3891 y_1		0.0739 $\Delta^2 y_0$		0.0008 $\Delta^4 y_{-1}$	
		0.7771 Δy_1		0.0078 $\Delta^3 y_0$		
2.1 x_2	8.1662 y_2		0.0817 $\Delta^2 y_1$			
		0.8588 Δy_2				
2.2 x_3	9.0250 y_3					

Gauss forward interpolation is

$$y = f(x) = f(x_0 + ph)$$

$$= y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{p(p-1)(p+1)}{3!}\Delta^3 y_{-1} + \frac{p(p-1)(p+1)(p-2)}{4!}\Delta^4 y_{-2}$$

$$+ \frac{(p+2)(p+1)p(p-1)(p-2)}{5!}\Delta^5 y_{-2}$$

$x = 1.91$

$$p = \frac{x - x_0}{h} = \frac{1.91 - 1.9}{0.1} = 0.1$$

$$= 6.6859 + (0.1)(0.7032) + \frac{(0.1)(0.1-1)}{2}(0.0669) + \frac{(0.1)(0.1-1)(0.1+1)}{3!}(0.007)$$

$$+ (0.1)(0.1-1)(0.1+1)$$

(2) Use gauss forward interpolation formula to find $f(3.3)$ from following table.

x	1	2	3	4	5
$y=f(x)$	15.30	15.10	15.00	14.50	14.00

Difference table-

x	$y=f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1	y_{-2} 15.30				
		Δy_{-2} −0.20			
2	y_{-1} 15.10		$\Delta^2 y_{-2}$ 0.10		
		Δy_{-1} −0.10		$\Delta^3 y_{-2}$ −0.50	
3	y_0 15.00		$\Delta^2 y_{-1}$ −0.40		$\Delta^4 y_{-2}$ 0.90
		Δy_0 −0.50		$\Delta^3 y_{-1}$ 0.40	
4	y_1 14.50		$\Delta^2 y_0$ 0		
		Δy_1 −0.50			
5	y_2 14.00				

$x = x_0 + ph$ $\quad h = 1,\ x_0 = 3,\ x = 3.3$

$$p = \frac{x - x_0}{h} = \frac{3.3 - 3}{1} = 0.3$$

Gauss forward interpolation:

$$y = f(x) = f(x_0 + ph) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{p(p+1)(p-1)}{3!}\Delta^3 y_{-1}$$

$$+ \frac{p(p+1)(p-1)(p-2)}{4!}\Delta^4 y_{-2} \cdots$$

$$= 15 + (0.3)(-0.50) + \frac{(0.3)(0.3-1)}{2}(-0.40) + \frac{(0.3+1)(0.3)(0.3-1)}{6}(0.40)$$

$$+ \frac{(0.3+1)(0.3)(0.3-1)(0.3-2)}{24}(0.90) = 14.89$$

$f(3.3) = 14.89$

③ find $y(25)$ given that $y_{20} = 24$, $y_{24} = 32$, $y_{28} = 35$, $y_{32} = 40$, using gauss forward difference formula

	x	y			
x_{-1}	20	24			
			Δy_{-1} 8		
x_0	24	32		$\Delta^2 y_{-1}$ -5	
			Δy_0 3		$\Delta^3 y_{-1}$ 7
x_1	28	35		2	
			5		
x_2	32	40			

$h = 4$, $x_0 = 24$, $x = 25$

$$p = \frac{x - x_0}{h} = \frac{25 - 24}{4} = 0.25$$

Gauss forward interpolation

$$y = f(x) = f(x_0 + ph)$$

$$= y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{p(p+1)(p-1)}{3!}\Delta^3 y_{-1}$$

$$= 32 + (0.25)(3) + \frac{(0.25)(0.25-1)}{2}(-5) + \frac{0.25(0.25+1)(0.25-1)}{6}(7)$$

$$f(25) = 32.945$$

④ Given $\sqrt{6500} = 80.6223$, $\sqrt{6510} = 80.6846$, $\sqrt{6520} = 80.7456$, $\sqrt{6530} = 80.8084$ find $\sqrt{6526}$ using gauss backward interpolation

	x	y			
x_{-3}	6500	y_{-3} 80.6223			
			0.0623		
x_{-2}	6510	y_{-2} 80.6846		-0.0013	
			0.061		0.0031
x_{-1}	6520	y_{-1} 80.7456		0.0018	
			0.0628		
x_0	6530	y_0 80.8084			

$x = 6526$, $h = 10$

$x_0 = 6530$

$$p = \frac{x - x_0}{h} = \frac{6526 - 6530}{10} = -0.4$$

$y = f(x) = f(x_0 + ph)$

$$= y_0 + p\Delta y_{-1} + \frac{p(p+1)}{2!}\Delta^2 y_{-1} + \frac{p(p-1)(p+1)}{3!}\Delta^3 y_{-2}$$

$$= 80.8084 + (-0.4)(0.0628) + \frac{(0.4)(-0.4+1)}{2}(0.0018)$$
$$+ \frac{(-0.4)(-0.4+1)(-0.4-1)}{6}(0.0031)$$

$f(2526) = 80.783$

(5) Using gauss backward difference formula find $y(8)$ from the table:

x	0	5	10	15	20	25
$f(x) = y$	7	11	14	18	24	32

	x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
y_{-2}	0	7					
			4 (Δy_{-2})				
y_{-1}	5	11		-1 ($\Delta^2 y_{-2}$)			
			3		2 ($\Delta^3 y_{-2}$)		
y_0	10	14		1		-1 ($\Delta^4 y_{-2}$)	
			4		1		0 ($\Delta^5 y_{-2}$)
y_1	15	18		2		-1	
			6		0 ($\Delta^3 y_0$)		
y_2	20	24		2 ($\Delta^2 y_1$)			
			8 (Δy_2)				
y_3	25	32					

$\Delta^3 y_1$

$y = f(x) = f(x_0 + ph)$

$$= y_0 + p\Delta y_{-1} + \frac{p(p+1)}{2!}\Delta^2 y_{-1} + \frac{p(p+1)(p-1)}{3!}\Delta^3 y_{-2} + \frac{p(p+1)(p+2)(p-1)}{4!}\Delta^4 y_{-2}$$

$$+ \frac{p(p+1)(p-1)(p+2)(p-2)}{5!}\Delta^5 y_{-3} + \cdots$$

$x = 8 \qquad x_0 = 10 \qquad h = 5$

$$p = \frac{x - x_0}{h} = \frac{8 - 10}{5} = \frac{-2}{5} = -0.4$$

$$y(8) = 14 + (-0.4)(3) + \frac{(-0.4)(-0.4+1)}{2}(1) + \frac{(-0.4)(-0.4+1)(-0.4-1)}{6}(2)$$

$$+ \frac{(-0.4)(-0.4+1)(-0.4+2)(-0.4-1)}{24}(-1)$$

$$+ \frac{(-0.4)(-0.4+1)(-0.4-1)(-0.4+2)(-0.4-2)}{120}(0)$$

$$\underline{y(8) = 12.76}$$

⑧ find $f(2.36)$ from the following table

x	1.6	1.8	2.0	2.2	2.4	2.6
y	4.95	6.05	7.39	9.03	11.02	13.48

	x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
x_{-4}	1.6	4.95					
			Δy_{-4} 1.1				
x_{-3}	1.8	6.05		$\Delta^2 y_{-4}$ 0.24			
			Δy_{-3} 1.34		$\Delta^3 y_{-4}$ 0.06		
x_{-2}	2.0	7.39		$\Delta^2 y_{-3}$ 0.3		$\Delta^4 y_{-4}$ -0.01	
			Δy_{-2} 1.64		$\Delta^3 y_{-3}$ 0.05		$\Delta^5 y_{-4}$ 0.06
x_{-1}	2.2	9.03		$\Delta^2 y_{-2}$ 0.35		$\Delta^4 y_{-3}$ 0.05	
			Δy_{-1} 1.99		$\Delta^3 y_{-2}$ 0.1		
x_0	2.4	11.02		$\Delta^2 y_{-1}$ 0.45			
			Δy_0 2.44				
x_1	2.6	13.48					

$h = 0.2$ $\qquad p = \dfrac{2.36 - 2.4}{0.2} = -0.2$

$x = 2.36$

$x_0 = 2.4$

Gauss backward interpolation

$$y = f(x) = f(x_0 + ph)$$

$$= y_0 + p\Delta y_{-1} + \frac{p(p+1)}{2!}\Delta^2 y_{-1} + \frac{p(p+1)(p-1)}{3!}\Delta^3 y_{-2} + \frac{p(p+1)(p+2)(p-1)}{4!}\Delta^4 y_{-2}$$

$$+ \frac{p(p+1)(p+2)(p-1)(p-2)}{5!}\Delta^5 y_{-3}$$

$$= 11.02 + (-0.2)(1.99) + \frac{(-0.2)(0.2+1)}{2}(0.45) + \frac{(-0.2)(-0.2+1)(-0.2-1)}{6}(0.1)$$

$$+ \frac{(0.2)(-0.2+1)(-0.2+2)(-0.2-1)}{24}(0) + 0$$

$y = 10.5892$

Lagranges interpolation (unequal intervals)

Let $y = f(x)$ be a polynomial of n^{th} degree, then Lagranges interpolation formula is given by.

$$y = f(x) = \frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)} f(x_0) +$$

$$\frac{(x-x_0)(x-x_2)(x-x_3)\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)\cdots(x-x_n)} f(x_1) + \cdots$$

$$+ \frac{(x-x_0)(x-x_1)\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1})} f(x_n)$$

Problem

① Evaluate $f(10)$ given $f(x) = 168, 192, 336$ at $x = 1, 7, 15$ respectively.

Given values can be written as.

	x_0	x_1	x_2
x	1	7	15
$y=f(x)$	168	192	336
	y_0	y_1	y_2

By Lagrange's interpolation formula.

$f(10) =$

$x = 10.$

$$y = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1)$$

$$+ \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2)$$

Given $x = 10$.

y at $x = 10$

$$y(10) = \frac{(10-7)(10-15)}{(1-7)(1-15)}(168) + \frac{(10-1)(10-15)}{(7-1)(7-15)}(192) +$$

$$\frac{(10-1)(10-7)}{(15-1)(15-7)}(336)$$

$$= \frac{3(-5)}{(-6)(-14)}(168) + \frac{9(-5)}{6(-8)}(192) + \frac{(9)(3)}{(14)(8)}(336)$$

$$= 237.$$

② A curve passes through points $(0,18)$, $(1,10)$, $(3,-18)$ and $(6,90)$ find slope of curve at $x=2$.

x	0	1	3	6
y	18	10	-18	90

Lagranges interpolation formula.

$$y = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}(y_0) + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}y_1 +$$

$$\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}y_3$$

$$y = \frac{(x-1)(x-3)(x-6)}{(0-1)(0-3)(0-6)}(18) + \frac{(x-0)(x-3)(x-6)}{(1-0)(1-3)(1-6)}(10) +$$

$$\frac{(x-0)(x-1)(x-6)}{(3-0)(3-1)(3-6)}(-18) + \frac{(x-0)(x-1)(x-3)}{(6-0)(6-1)(6-3)}90.$$

$$= \frac{(x-1)(x-3)(x-6)}{(-1)(-3)(-6)}(-18) + \frac{x(x-3)(x-6)}{(1)(-2)(-5)}(10) +$$

$$\frac{x(x-1)(x-6)}{(3)(2)(-3)}(-18) + \frac{x(x-1)(x-3)}{(6)(5)(3)} 90.$$

$$= -\left[(x^2-3x-x+3)(x-6) + (x^2-3x)(x-6) + (x^2-x)(x-6) + (x^2-x)(x-3)\right]$$

$$= -[(x^2-4x+3)(x-6) + x^3-6x^2-3x^2 \cancel{\ } -18x + x^3-6x^2 -x^2+6x+x^3-3x^2-x^2+3x]$$

$$= -[x^3-6x^2-4x^2+24x+3x-18] + 3x^3-20x^2+27x.$$

$$= -x^3+10x^2-\cancel{27x}+18+3x^3-20x^2+\cancel{27x}$$

$$y = 2x^3-10x^2+18.$$

Slope of curve.

$$y = 2x^3-10x^2+18.$$

$$y' = 6x^2-20x.$$

at $(x=2)$ $y' = 6(4)-20(2)$

$$= 24-40$$

$$= -16.$$

(3) find parabola passing through points (0,1), (1,3) and (3, 55) using lagrange's interpolation.

	x_0	x_1	x_2
x	0	1	3
y	1	3	55
	y_0	y_1	y_2

$$y = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1) + \cdots$$

$$\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2).$$

$$= \frac{(x-1)(x-3)}{(0-1)(0-3)}(1) + \frac{(x-0)(x-3)}{(1-0)(1-3)}(3) +$$

$$\frac{(x-0)(x-1)}{(3-0)(3-1)}(55)$$

$$= \frac{x^2-4x+3}{3} + \left(\frac{-3x^2+9x}{+2}\right) + \frac{55x^2-55x}{6}$$

$$= \frac{2x^2-8x+6-9x^2+27x+55x^2-55x}{6}$$

$$= \frac{48x^2-36x+6}{6} = 8x^2-6x+1$$

$$y = 8x^2-6x+1.$$

$$y = 8\left(x^2-\frac{3}{4}x\right)+1$$

$a = 8 \quad b = -6 \quad \pm\left(\frac{b}{2a}\right)^2$

$$y = 8\left(x^2-\frac{3}{4}x+\left(\frac{9}{64}\right)-\left(\frac{9}{64}\right)\right)+1$$

$$y = 8\left(\left(x-\frac{3}{8}\right)^2 - \frac{9}{64}\right) + 1.$$

$$y = 8\left(x-\frac{3}{8}\right)^2 - \frac{9}{8} + 1$$

$$\boxed{y = 8\left(x-\frac{3}{8}\right)^2 - \frac{1}{8}}$$

(4) Using lagranges interpolation formula, find y(10) from following table.

	x_0	x_1	x_2	x_3
x	5	6	9	11
y	12	13	14	16
	y_0	y_1	y_2	y_3

lagrange's interpolation formula.

$$y = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}(y_0) + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}y_3$$

$$= \frac{(x-6)(x-9)(x-11)}{(5-6)(5-9)(5-11)}(12) + \frac{(x-5)(x-9)(x-11)}{(6-5)(6-9)(6-11)}(13)$$

$$+ \frac{(x-5)(x-6)(x-11)}{(9-5)(9-6)(9-11)}(14) + \frac{(x-5)(x-6)(x-9)}{(11-5)(11-6)(11-9)}(16)$$

$$= \frac{(x^3-26x^2+219x-594)}{-2} + \frac{(x^3-25x^2+209x-495)13}{15}$$

$$+ \frac{(x^3-22x^2+151x-330)7}{12} + \frac{(x^3-20x^2+129x-270)4}{15}$$

$$y = \frac{\cancel{4}^{2} \times 1 \times 11}{\cancel{2}} - \frac{\cancel{5}}{\cancel{15}_{3}}(13)^{4.33} + \frac{5 \times \cancel{4} \times 1 \times 7}{12_{3}} + \frac{5 \times 4 \times 1 \times 4}{15}$$

$$y = 2 - 4.33 + 11.66 + 5.33$$

$$\boxed{y = 14.66}$$

Curve fitting

Method of least squares.

① Fitting a straight line.

Let $y = a + bx$ is a straight line to be fitted to the given data, then the normal equations are.

$$\Sigma y = ma + b\Sigma x \quad —②$$

$$\Sigma xy = a\Sigma x + b\Sigma x^2 \quad —③$$

Solving 2 & 3 equations for a, b we get required straight line of best fit.

② Fitting a parabola:- (second polynomial)

Let $y = a + bx + cx^2$ —① is a parabola to be fitted to the given data, The normal equations are.

$$\Sigma y = ma + b\Sigma x + c\Sigma x^2 \quad —②$$

$$\Sigma xy = a\Sigma x + b\Sigma x^2 + c\Sigma x^3 \quad —③$$

$$\Sigma x^2 y = a\Sigma x^2 + b\Sigma x^3 + c\Sigma x^4 \quad —④$$

Solving ②, ③ & ④ equations for a, b, c, we get required parabola of best fit.

Problems

1. By the method of least squares, find the straight line that best fit the following data.

x	1	2	3	4	5
y	14	27	40	55	68

Sol

x	y	xy	x^2
1	14	14	1
2	27	54	4
3	40	120	9
4	55	220	16
5	68	340	25

$\Sigma x = 15$ $\Sigma y = 204$ $\Sigma xy = 748$ $\Sigma x^2 = 55$.

Let $y = a + bx$. ——① .

Normal equations are

$\Sigma y = ma + b\Sigma x$ ——② .

$\Sigma xy = a\Sigma x + b\Sigma x^2$ ——③ .

$204 = 5a + b(15)$ ——④

$748 = 15a + 55b$ ——⑤ .

Solving ④ & ⑤ .

$a = 0$, $b = 13.6$.

Substitute a, b values in ①

$\boxed{y = 13.6x}$.

(2) Fit a st line is following data.

x	6	7	7	8	8	8	9	9	10
y	5	5	4	5	4	3	4	3	3

x	y	xy	x^2
6	5	30	36
7	5	35	49
7	4	28	49
8	5	40	64
8	4	32	64
8	3	24	64
9	4	36	81
9	3	27	81
10	3	30	100
$\Sigma x=72$	$\Sigma y=36$	$\Sigma xy=282$	$\Sigma x^2=588$

let $y = a+bx$ — (1)

Normal equations are

$\Sigma y = ma + b\Sigma x$ — (2)

$\Sigma xy = a\Sigma x + b\Sigma x^2$ — (3)

$36 = 9(a) + b(72)$ — (4)

$282 = a(72) + b(588)$ — (5)

solving ④ & ⑤.

$a = 8 \quad b = -0.5.$

$$\boxed{y = -0.5x + 8}$$

③ A chemical company wishing to study the effect of extraction time on the efficiency of an extraction operation obtained the data shown in following data.

Extraction time in min. x	27	45	41	19	3	39	19	49	15	31
efficiency y	57	64	80	46	62	72	52	77	57	68

fit a st line

Let $y = a + bx$ — ①.

x	y	xy	x^2
27	57	1539	729
45	64	2880	2025
41	80	3280	1681
19	46	874	361
3	62	186	9
39	72	2808	1521
19	52	988	361
49	77	3773	2401
15	57	855	225
31	68	2108	961
$\Sigma x = 288$	$\Sigma y = 635$	$\Sigma xy = 19291$	$\Sigma x^2 = 10274$

Normal eqns are:

$\Sigma y = am + b\Sigma x$ — ②

$\Sigma xy = a\Sigma x + b\Sigma x^2$ — ③

$635 = 10a + 288b$ — ④

$19291 = 288a + 10274b$ — ⑤

solving ④ & ⑤

$a = 48.90 \quad b = 0.5066.$

$y = 48.90 + (0.5066)x.$

④ Fit st line to form $y = a + bx$. for following.

x	0	5	10	15	20	25
y	12	15	17	22	24	30

x	y	xy	x^2
0	12	0	0
5	15	75	25
10	17	170	100
15	22	330	225
20	24	480	400
25	30	750	625

$\Sigma x = 75$ $\quad \Sigma xy = 1805$

$\Sigma y = 120$ $\quad \Sigma x^2 = 1375$

let $y = a + bx$

Normal equations are

$\Sigma y = ma + b\Sigma x$

$\Sigma xy = a\Sigma x + b\Sigma x^2$

$120 = 6a + b(75)$ — ①

$1805 = a(75) + b(1375)$ — ②

Solving ① & ②.

$a = 11.28$ $\quad b = 0.69$

$y = 11.28 + (0.69)x$

(5) Fit a second degree polynomial for following data by the method of least squares.

x	0	1	2	3	4
y	1	1.8	1.3	2.5	6.3

Second degree polynomial is

$$y = a + bx + cx^2 \quad (2)$$

Normal eq are

$$\Sigma y = na + b\Sigma x + c\Sigma x^2$$

$$\Sigma xy = a\Sigma x + b\Sigma x^2 + c\Sigma x^3$$

$$\Sigma x^2 y = a\Sigma x^2 + b\Sigma x^3 + c\Sigma x^4$$

x	y	xy	x^2y	x^2	x^3	x^4
0	1	0	0	0	0	0
1	1.8	1.8	1.8	1	1	1
2	1.3	2.6	5.2	4	8	16
3	2.5	7.5	22.5	9	27	81
4	6.3	25.2	100.8	16	64	256
$\Sigma x = 10$	$\Sigma y = 12.9$	$\Sigma xy = 37.1$	$\Sigma x^2 y = 130.3$	$\Sigma x^2 = 30$	$\Sigma x^3 = 100$	$\Sigma x^4 = 354$

$$12.9 = 5(a) + b(10) + c(30)$$

$$37.1 = a(10) + b(30) + c(100)$$

$$130.3 = a(30) + b(100) + c(354)$$

$a = 1.42 \quad b = -1.07 \quad c = 0.55$

$$0.55x^2 - 1.07x + 1.42$$

⑥ By the method of least squares fit a parabola, of the form.

x	2	4	6	8	10
y	3.07	17.85	31.47	57.38	91.29

x	y	xy	x^2y	x^2	x^3	x^4
2	3.07	6.14	12.28	4	8	16
4	12.85	51.4	205.6	16	64	256
6	31.47	188.82	1132.92	36	216	1296
8	57.38	459.04	3672.32	64	512	4096
10	91.29	912.9	9129	100	1000	10000
30	196.06	1618.3	14152.12	220	1800	15664

$$196.06 = 5a + 30b + 220c$$

$$1618.3 = 30a + 220b + 1800c$$

$$14152.12 = 220a + 1800b + 15664c$$

$$a = 0.696$$

$$b = -0.855$$

$$c = 0.991$$

eqn of parabola.

$$y = 0.696 - 0.855x + 0.991x^2$$

Fit a parabola of the form $y = a + bx + cx^2$ to the following data

x	1	2	3	4	5	6	7.
y	2.3	5.2	9.7	16.5	29.4	35.5	54.4.

x	y	xy	x^2	x^2y	x^3	x^4
1	2.3	2.3	1	2.3	1	1
2	5.2	10.4	4	20.8	8	16
3	9.7	29.1	9	87.3	27	81
4	16.5	66.0	16	264.0	64	256
5	29.4	147.0	25	735.0	125	625
6	35.5	213.0	36	1278	216	1296
7	54.4	380.8	49	2665.6	343	2401
28	153.0	848.6	140	5053	784	4676

$153 = 7a + 28b + 140c$

$848.6 = 28a + 140b + 784$

$5053 = 140a + 784b + 4676c$

$a = 2.3714 \quad b = -1.0928 \quad c = 1.1928$

$y = 1.1928x^2 - (1.0928)x + 2.3714$

Non lineal curves.

power curve.

1. let $y = a \cdot e^{bx}$ — ①.

Taking $\log_e$ on both sides.

$\log_e y = \log_e a e^{bx}$.

$\log_e y = \log_e a + \log_e e^{bx}$.

$\log_e y = \log_e a + bx$.

$y = A + bx$.

Now the normal equations are

$\Sigma y = mA + b\Sigma x$ — ②

$\Sigma xy = A\Sigma x + b\Sigma x^2$ — ③.

Solving ②, ③ equations to get A & b.

since $A = \log_e a$

$a = e^A$.

problem :- Find the curve of best fit of the type $y = a \cdot e^{bx}$. to the following data by the method of least squares.

x	1	5	7	9	12
y	10	15	12	15	21

Given $y = ae^{bx}$ — ① [$\because \log_e = \ln$]

take $\log_e$ on both sides.

$\log_e y = \log_e a + bx$.

$y = A + bx$.

Normal equations are.

$\Sigma y = mA + b\Sigma x$ — ①

$\Sigma xy = A\Sigma x + b\Sigma x^2$ — ③.

x.	y	Y = ln y.	xy	x^2.
1	10	2.3025.	2.3025.	1
5	15.	2.7080.	13.54.	25.
7	12.	2.4849.	17.3943.	49.
9	15.	2.7080	24.372.	81
12.	21.	3.0445.	36.534.	144.
34	73	13.249	94.1428.	300.

sub $\Sigma x, \Sigma x^2, \Sigma Y, \Sigma xy$ values in ②&③.

$13.249 = 5A + 34b$ — ④.

$94.1428 = 34A + 300b$ — ⑤.

solving ④ & ⑤.

$A = 2.2495 \quad b = 0.0588.$

$a = e^A = e^{2.2495} = 9.4829.$

substitute a & b in ①.

Hence $y = 9.4829\, e^{(0.0588)x}$.

② find the curve of best fit of type $y = a \cdot e^{bx}$ to the following data by the method of least squares.

x	0.0	0.5	1.0	1.5	2.0	2.5
y	0.10	0.45	2.15	9.15	40.35	180.75

x	y	$Y = \ln y$	~~xy~~	xY	x^2
0.0	0.10	−2.3025	~~0~~	0	0
0.5	0.45	−0.7985	~~0.2125~~	−0.39925	0.25
1.0	2.15	0.7654	~~2.15~~	0.7654	1
1.5	9.15	2.2137	~~20.5875~~	3.3205	2.25
2.0	40.35	3.6975		7.395	4
2.5	180.75	5.1971		12.9927	6.25
7.5	232.85	8.7727		24.0744	13.75

$$8.7727 = 6A + 7.5b$$

$$24.0744 = 7.5A + 13.75b$$

$$A = -2.2831$$

$$b = 2.9962$$

$$a = e^A = 0.1019$$

$$y = A + bx$$

$$= 0.1019 + e^{(2.9962)x}$$

(3) Fit the curve $y = a \cdot e^{bx}$.

x	0	1	2	3	4	5	6	7	8
y	20	30	52	77	135	211	326	550	1052

$$y = 18.95 \cdot e^{0.486x}.$$

x	y	$Y = \ln y$	xY	x^2
0	20	2.9957	0	0
1	30	3.4011	3.4011	1
2	52	3.9512	7.9024	4
3	77	4.3438	13.0314	9
4	135	4.9052	19.6208	16
5	211	5.3518	26.759	25
6	326	5.7868	34.7208	36
7	550	6.3099	44.1693	49
8	1052	6.9584	55.6672	64
36		44.0039	205.272	204

Normal equations are.

$$\Sigma Y = mA + b\Sigma x.$$

$$\Sigma xY = A\Sigma x + b\Sigma x^2.$$

$$44.0039 = 9A + b(36)$$

$$205.272 = A(36) + b(204)$$

$$A = 2.9388 \qquad b = 0.4876.$$

$$a = e^A = 18.8931$$

$$y = a \cdot e^{bx} = (18.8931) e^{(0.4876)x}.$$

② $y = ab^x$.

Taking $\log_{10}$ on both sides.

$\log_{10} y = \log_{10} (ab^x)$

$\log_{10} y = \log_{10} a + x \log_{10} b$

$\log y = \log a + x \log b$.

$Y = A + Bx$ — ②

Now normal equations of ②

$\Sigma Y = mA + B\Sigma x$ — ③

$\Sigma xY = A\Sigma x + B\Sigma x^2$ — ④.

Solving ③ & ④ for A & B.

$A = \log_{10} a \Rightarrow a = 10^A$.

$B = \log_{10} b \Rightarrow b = 10^B$.

Sub A & B in ① to get required curve $y = ab^x$.

<u>Problems:-</u> ① Fit $y = ab^x$ by method of least squares to following data.

x	0	1	2	3	4	5	6	7.
y	10	21	35	59	92	200	400	610.

<u>Sol</u> To fit a curve $y = a.b^x$ — ①.

Taking $\log_{10}$ on both sides

$\log_{10} y = \log_{10} a + x \log_{10} b$.

$Y = A + Bx$ — ②

Normal equations are.

$\Sigma Y = mA + B\Sigma x$ — ③

$\Sigma xY = A\Sigma x + B\Sigma x^2$ — ④.

x	y	$Y = \log_{10} y$	xY	x^2
0	10	1	0	0
1	21	1.3222	1.3222	1
2	35	1.5440	3.088	4
3	59	1.7708	5.3124	9
4	92	1.9637	7.8548	16
5	200	2.3010	11.505	25
6	400	2.6020	15.612	36
7	610	2.7853	19.4971	49
28	1427	15.289	64.1915	140

$15.289 = 8A + B(28)$

$64.1915 = A(28) + B(140)$

$A = 1.0211 \Rightarrow a = 10^{1.0211} = 10.4978$

$B = 0.2542 \Rightarrow b = 10^{0.2542} = 1.7955$

$y = (10.4978)(1.7955)^x$

② Fit curve of form

$y = ab^x$

x	77	100	185	239	285
y	2.4	3.4	7.0	11.1	19.6

To fit a curve $y = ab^x$

log on both sides

$\log_{10} y = \log_{10} a + x \log_{10} b$

$Y = A + Bx$

Normal equations are

$\Sigma Y = mA + B(\Sigma x)$

$\Sigma xY = A \cdot \Sigma x + B(\Sigma x^2)$

x	y	$Y = \log_{10} y$	xY	x^2
77	2.4	0.3802	29.2754	5929
100	3.4	0.5314	53.14	10000
185	7.0	0.8450	156.325	34225
239	11.1	1.0453	249.8267	57121
285	19.6	1.2922	368.277	81225
886		4.0941	856.7941	188500

$$4.0941 = 5A + B(886)$$

$$856.7941 = 886A + B(188500)$$

$$A = 0.0801 \qquad B = 0.00416$$

$$a = 10^{0.0801} \qquad b = 10^{0.00416}$$

$$a = 1.2025 \qquad b = 1.0096$$

$$y = (1.2025)(1.0096)^{x}$$

* (3) Fitting a curve $y = ax^{b}$ — (1)

Taking $\log_{10}$ on both sides

$$\log_{10} y = \log_{10} a x^{b}$$

$$\log_{10} y = \log_{10} a + b \log_{10} x$$

$$Y = A + bX \text{ — (2)}$$

write normal eqn of (2)

$$\Sigma Y = mA + b\Sigma X \text{ — (3)}$$

$$\Sigma XY = A\Sigma X + b\Sigma X^2 \text{ — (4)}$$

solving (3) & (4) $A = \log_{10} a$ $\quad a = 10^{A}$

subs a, b in (1) to get required curve

① Fit a power curve of form $y = ax^b$ for following

x	1	2	4	6
y	6	4	2	2.

Fit curve of $y = ax^b$ — ①

$\log_{10} y = \log_{10} a + b \log_{10} x$

$Y = A + bX$ — ②

Normal eqns are-

$\Sigma Y = mA + b\Sigma X$ — ③.

$\Sigma YX = A\Sigma X + b\Sigma X^2$ — ④.

x	y	$Y = \log_{10} y$	$X = \log_{10} x$	XY	X^2
1	6	0.7781	0	0	0
2	4	0.6020	0.3010	0.1812	0.0906
4	2	0.3010	0.6020	0.1812	0.3624
6	2	0.3010	0.7781	0.2342	0.6054
		$\Sigma Y = 1.9821$	$\Sigma X = 1.6811$	$\Sigma XY = 0.5966$	$\Sigma X^2 = 1.0584$

substitute values in eq ③ & ④.

$1.9821 = 4A + b(1.6811)$ — ③

$0.5966 = A(1.6811) + b(1.0584)$ — ④

Solving ③ & ④ ~~[illegible]~~ ~~[illegible]~~.

~~[illegible]~~

~~[illegible]~~

$A = 0.7779 \quad b = -0.6719.$

$a = 10^A = 10^{0.7779} = 5.9965.$

substitute a, b values in ①.

$y = (5.9965)\, x^{-0.6719}.$

(2) Fit a curve $y = a x^b$ to following data

x	1	2	3	4	5	6.
y	2.98	4.26	5.21	6.10	6.80	7.50.

x	y.	$Y = \log_{10} y$	$X = \log_{10} x$	XY	X^2.
1	2.98	0.4742.	0	0	0
2	4.26	0.6294.	0.3010.	0.1894	0.0906
3	5.21	0.7168	0.4771	0.3419	0.2276
4	6.10	0.7853	0.6020	0.4727.	0.3624
5	6.80	0.8325	0.6989	0.5818.	0.4884
6.	7.50	0.8750	0.7781	0.6808.	0.6054.
		4.3132	2.8571	2.256	1.7744.

$$4.3132 = 6A + 2.8571b.$$

$$2.266 = 2.8571A + 1.7744b.$$

$$A = 0.4748$$

$$b = 0.5125.$$

$$a = 10^A = 10^{0.4748} = 2.9908.$$

$$y = 2.9908\ x^{0.5125}.$$

Numerical Techniques.

Topic Algebraic & Transcendental equation.

1. Polynomial function :- A function $f(x)$ is said to be a polynomial function if $f(x)$ is a polynomial in x.

i.e $f(x) = a_0x^n + a_1x^{n-1} \cdots + a_{n-1}x + a_n$, where $a_0 \neq 0$. the coefficients $a_0, a_1, \cdots a_n$ are real constants and n is non-negative integer.

2. Algebraic function :- A function which is a sum or difference or product of two polynomials is called an algebraic function, otherwise, the function is called a transcendental or non-algebraic function.

Ex :- ① $f(x) = c_1e^x + c_2e^{-x}$, $f(x) = e^{5x} - \frac{x^3}{2} + 3 = 0$.

are examples of transcendental equations.

② $f(x) = x^3 - 2x + 1 = 0$; $f(x) = x^2 + 1 = 0$ are the examples of algebraic function.

Bisection method :- Bisection method is a simple iteration method to be solved in an equation

This method is also known as BOLZANO method of successive Bisection.

Suppose, equation of form $f(x) = 0$ has exactly one real root between two real numbers x_0, x_1.

1. Find a root of equation $x^3-5x+1=0$ using bisection method in 5 stages.

Given $f(x)=0$

$\Rightarrow x^3-5x+1=0.$

Use trial and error method.

$x=0, \quad f(0)=1>0$

$x=1, \quad f(1)=1-5+1$

$= 2-5 = -3<0.$

Root lies b/w 0 & 1.

let $x_0=0, \; x_1=1$ +ve ———— −ve, x_0 x_1

By bisection method

first approximation is given by

$$x_2 = \frac{x_0+x_1}{2} = \frac{1}{2} = 0.5.$$

$f(x_2) = f(0.5) = (0.5)^3 - 5(0.5) + 1$

$= -1.375 < 0$ $\quad$ $-ve$ $\quad$ $+ve$

x_2 $\quad$ x_0

Root lies b/w x_2 & x_0.

second approximation is

$$x_3 = \frac{x_2 + x_0}{2} = \frac{0.5 + 0}{2} = 0.25$$

$f(x_3) = f(0.25) = (0.25)^3 - 5(0.25) + 1.$

$= -0.2343 < 0$ $\quad$ $-ve$ $\quad$ $+ve$

x_3 $\quad$ x_0.

Root lies b/w x_3 & x_0.

Third approximation is $x_4 = \frac{x_3 + x_0}{2} = \frac{0.25 + 0}{2} = 0.1$

$f(x_4) = f(0.125) = (0.125)^3 - 5(0.125) + 1.$

$= 0.3769 > 0.$ $\quad$ $+ve$ $\quad$ $-ve$

x_4 $\quad$ x_3

Root lies b/w x_2 & x_3

fourth approx is

$$x_5 = \frac{x_3 + x_4}{2} = \frac{0.25 + 0.125}{2} = 0.1875$$

$f(x_5) = f(0.1875)$

$= (0.1875)^3 - 5(0.1875) + 1$

$= 0.0690 > 0$ $\quad$ $+ve$ $\quad$ $-ve$

x_5 $\quad$ x_3.

Root lies b/w x_5 & x_3.

Fifth approximation is

$x_6 = 0.1875 + 0.25/2.$

$x_6 = 0.2187$

The root of $x^3 - 5x + 1$ is 0.2187

2) Find a positive root of $x^3 - x = 1$ using bisection method.

let $f(x) = x^3 - x - 1 = 0$.

By trail & error method.

$x = 0 \quad f(0) = -1 < 0.$

$x = 1 \quad f(1) = 1^3 - 1 - 1 = 0.$

$= -1 < 0.$

$x = 2 \quad f(2) = 8 - 2 - 1 = 5.$

$= 5 > 0.$

root lies b/w 1 and 2

$x_0 = 1 \quad x_1 = 2.$ +ve 2, -ve 1.

first approximations $x_2 = \frac{x_0 + x_1}{2} = \frac{1+2}{2} = 1.5.$

$f(x_2) = f(1.5) = (1.5)^3 - 1.5 - 1 = 0.875.$

root lies b/w x_2 & x_0. +ve x_2, -ve x_0.

Second approximation.

$x_0 = 1 \quad x_2 = 1.5.$

$x_3 = \frac{1+1.5}{2} = \frac{2.5}{2} = 1.25.$

$\boxed{x_3 = 1.25}$

$f(x_3) = -0.2968 < 0,$ (-) x_3, (+) x_2.

Root lies b/w x_3 & x_2.

Third approximation is $x_4 = \frac{x_3 + x_2}{2} = \frac{1.25+1.5}{2}.$

$= 1.375.$

$\boxed{x_4 = 1.375}$

$f(x_4) = 0.2246 > 0$ $\quad$ $\underset{x_4}{(+)}$ —— $\underset{x_3}{(-)}$

Root lies b/w x_4 & x_3.

fourth approximation $x_5 = \frac{x_4 + x_3}{2}$.

$$x_5 = \frac{1.375 + 1.25}{2} = 1.3125$$

$\boxed{x_5 = 1.3125}$

$f(x_5) = -0.051 < 0$ $\quad$ $\underset{x_5}{(-)}$ —— $\underset{x_4}{(+)}$

Root lies b/w x_5 & x_4.

fifth approximation is $x_6 = \frac{x_5 + x_4}{2}$.

$$= \frac{1.3125 + 1.375}{2}.$$

$$x_6 = 1.3437.$$

$f(x_6) = 0.0826 > 0$ $\quad$ $\underset{x_6}{+ve}$ —— $\underset{x_5}{-ve}$

Root lies b/w x_6 & x_5

Sixth approximation is $x_7 = \frac{x_6 + x_5}{2}$.

$$x_7 = \frac{1.3437 + 1.3125}{2} = 1.3281$$

$$x_7 = 1.3281$$

$f(x_7) = 0.0144 > 0$ $\quad$ $\underset{x_7}{+ve}$ —— $\underset{x_5}{-ve}$

Root lies b/w x_7 & x_5

seventh approximation $x_8 = \frac{x_7 + x_5}{2} = \frac{1.3281 + 1.3125}{2}$

$$x_8 = 1.3203.$$

$f(x_8) = -0.0187 < 0.$ $\quad$ $-ve \quad +ve.$ $\quad x_8 \quad x_7.$

$\therefore$ Root lies b/w x_8 & x_7.

eighth approximation is -

$$x_9 = \frac{x_8 + x_7}{2} = \frac{1.3203 + 1.3281}{2}$$

$$x_9 = 1.3242.$$

$f(x_9) = f(1.3242) = -0.002 < 0.$ $\quad - \quad + \quad x_9 \quad x_7.$

Root lies b/w x_9 & x_7.

9th approximation is

$$x_{10} = \frac{x_9 + x_7}{2}.$$

$$x_{10} = 1.3261.$$

$f(x_{10}) = 0.006 > 0.$ $\quad (+) \quad (-) \quad x_{10} \quad x_9.$

Root lies b/w x_{10} & x_9.

10th approximation is

$$x_{11} = \frac{x_{10} + x_9}{2} = \frac{1.3261 + 1.3242}{2}.$$

$$x_{11} = 1.3251$$

$f(x_{11}) = 0.0016 > 0.$ $\quad + \quad - \quad x_{11} \quad x_9.$

11th approximation is

$$x_{12} = \frac{x_{11} + x_9}{2} = \frac{1.325 + 1.3242}{2}.$$

$$\boxed{x_{12} = 1.3246}$$

$f(x_{12}) = -0.0005 < 0$. $\quad \underset{x_{12}}{(-)} \longrightarrow \underset{x_{11}}{(+)}$

$\therefore$ Root lies b/w x_{12} & x_{11}

12^{th} approximation.

$$x_{13} = \frac{x_{12} + x_{11}}{2} = 1.3248.$$

$$x_{13} = 1.3248.$$

$f(13) = 0.0003 > 0.$

root lies b/w x_{13} & x_{12}.

13^{th} approximation.

$$x_{14} = \frac{x_{13} + x_{12}}{2}.$$

$$x_{14} = 1.3247.$$

$f(x_{14}) = -0.0007 < 0.$ $\quad \underset{x_{14}}{-} \longrightarrow \underset{x_{13}}{+}$

14^{th} approximation.

$$x_{15} = \frac{x_{14} + x_{13}}{2} = \frac{1.3247 + 1.3248}{2}.$$

$$x_{15} = 1.3247.$$

Q. Find real root of equation x^3-6x-4 by bisection method.

Let $f(x) = x^3 - 6x - 4 = 0$.

By trial & error method.

$x=0$, $f(0) = -4 < 0$.

$x=1$, $f(1) = -9 < 0$.

$x=2$, $f(2) = -8 < 0$.

$x=3$, $f(3) = 27-18-4 = 5 > 0$.

$\therefore$ Root lies b/w 2 & 3.

Let $x_0 = 2$, $x_1 = 3$ $\quad \overset{-}{x_0}$ —— $\overset{+}{x_1}$

1st app. is $x_2 = \dfrac{x_0+x_1}{2} = \dfrac{2+3}{2} = 2.5$.

$f(x_2) = f(2.5) = (2.5)^3 - 6(2.5) - 4 = -3.375 < 0$

root lies b/w x_2 & x_1.

2nd app. is

$$x_3 = \frac{x_2+x_1}{2} = \frac{2.5+3}{2} = \frac{5.5}{2}.$$

$$x_3 = 2.75.$$

$f(x_3) = f(2.75) = 0.2968 > 0$.

root lies b/w x_2 & x_3 $\quad \overset{+}{x_3}$ —— $\overset{-}{x_2}$

3rd app is

$$x_4 = \frac{x_2+x_3}{2} = \frac{2.5+2.75}{2} = 2.625.$$

$$x_4 = 2.625.$$

$f(x_4) = f(2.625) = -1.6621$

root lies b/w x_4 and x_3.

4^{th} app is $x_5 = \frac{x_4 + x_3}{2} = \frac{2.625 + 2.75}{2}$

$x_5 = 2.6875$

root lies b/w x_5 & x_3 $\quad \underset{x_5}{(-)} \overline{\quad\quad\quad} \underset{x_3}{(+)}$

5^{th} app is $x_6 = \frac{x_5 + x_3}{2} = \frac{2.6875 + 2.75}{2}$

$x_6 = 2.7187$

$f(x_6) = f(2.7187) = -0.2173 < 0$ $\quad \underset{x_6}{-} \overline{\quad\quad\quad} \underset{x_3}{+}$

root lies b/w x_6 & x_3

6^{th} app is $x_7 = \frac{x_6 + x_3}{2}$

$x_7 = 2.7343$

$f(x_7) = f(2.7343) = 0.0369 > 0$ $\quad \underset{x_7}{+} \overline{\quad\quad\quad} \underset{x_6}{-}$

root lies b/w x_7 & x_6

7^{th} app is $x_8 = \frac{x_7 + x_6}{2} = 2.7265$

$x_8 = 2.7265$

$f(x_8) = f(2.7265) = -0.0907 < 0$ $\quad \underset{x_8}{-} \overline{\quad\quad\quad} \underset{x_7}{+}$

root lies b/w x_9 & x_7

9^{th} app is $x_{10} = \frac{x_9 + x_7}{2} = 2.7323$

$x_{10} = 2.7323$

$f(x_{10}) = f(2.7323) = 0.004 > 0$ $\quad \underset{x_{10}}{+} \overline{\quad\quad\quad} \underset{x_9}{-}$

root lies b/w x_{10} & x_9.

10^{th} app is $x_{11} = \frac{x_{10} + x_9}{2} = 2.7323.$

$x_{11} = 2.7323.$

$\therefore$ 2.7323 is root of eqn $x^3 - 6x - 4$.

Q. Find a real root of the equation $x \log_{10} x = 1.2$ by bisection method.

$f(x) = x \log x - 1.2 = 0.$

By trial and error method.

$x = 1 \quad f(x) = 1 \log 1 - 1.2 = -1.2 < 0.$

$x = 2 \quad f(2) = 2 \log 2 - 1.2 = -0.5979 < 0.$

$x = 3 \quad f(3) = 3 \log 3 - 1.2 = 0.2313 > 0$

$\therefore$ root lies b/w 2 and 3.

$x_0 = 2 \quad x_1 = 3.$

first approximation.

$$x_2 = \frac{x_0 + x_1}{2} = \frac{2+3}{2} = \frac{5}{2} = 2.5$$

$\boxed{x_2 = 2.5}$

$f(2.5) = 2.5 \log 2.5 - 1.2 = -0.2051 < 0.$

Root lies b/w x_1 and x_2

2^{nd} approximation

$$x_3 = \frac{x_1 + x_2}{2} = \frac{2.5 + 3}{2} = \frac{5.5}{2} = 2.75$$

$\boxed{x_3 = 2.75}$

$f(2.75) = 2.75 \log 2.75 - 1.2 = 0.008 > 0$ $\quad \overset{+}{x_3} \text{———} \overset{(-)}{x_2}$

third approximation.

$x_4 = \frac{x_3 + x_2}{2} = \frac{2.75 + 2.5}{2} = \frac{5.25}{2} = 2.625$

$\boxed{x_4 = 2.625}$

$f(2.625) = f(x_4) = -0.0997 < 0$ $\quad \overset{-}{x_4} \text{———} \overset{+}{x_3}$

fourth approximation

$x_5 = \frac{x_4 + x_3}{2} = \frac{2.75 + 2.625}{2} = \frac{5.375}{2}$

$\boxed{x_5 = 2.6875}$

$f(2.8875) = f(x_5) = -0.0461$ $\quad \overset{-}{x_5} \text{———} \overset{+}{x_3}$

fifth approximation.

$x_6 = \frac{x_5 + x_3}{2} = \frac{2.6875 + 2.75}{2} =$

$\boxed{x_6 = 2.71875}$ $\quad \overset{-}{x_6} \text{———} \overset{+}{x_3}$

$f(x_6) = f(2.7187) = -0.0191$

sixth approximation.

$x_7 = \frac{x_6 + x_3}{2} = \frac{2.7187 + 2.75}{2} = 2.7343$

$\boxed{x_7 = 2.7343}$ $\quad \overset{+}{x_3} \text{———} \overset{-}{x_7}$

$f(x_7) = f(2.7343) = -0.0055$

seventh app.

$x_8 = \frac{x_7 + x_3}{2} = \frac{2.7343 + 2.75}{2} = 2.7421$

$x_8 = 2.7421$

$f(x_8) = 0.0012 > 0$ $\quad \overset{-}{x_7} \text{———} \overset{+}{x_8}$

eigth app.

$x_9 = \frac{x_7 + x_8}{2} = 2.7382$

$f(x_9) = f(2.7382) = -0.002150$

~~ninth ninth~~ ninth app.

roots lies b/w x_9 & x_8. $\overset{-}{x_9}$ —— $\overset{+}{x_8}$

$$x_{10} = \frac{x_9 + x_8}{2} = \frac{2.7382 + 2.7421}{2}$$

$$\boxed{x_{10} = 2.7401}$$

$f(x_{10}) = f(2.7401) = -0.00047 < 0$. $\overset{-}{x_{10}}$ —— $\overset{+}{x_8}$

10^{th} approxi.

roots lies b/w x_{10} & x_8.

$$x_{11} = \frac{x_{10} + x_8}{2} = \frac{2.7401 + 2.7421}{2}$$

$$\boxed{x_{11} = 2.7411}$$

$f(x_{11}) = f(2.7411) = 0.00039 > 0$. $\overset{-}{x_{10}}$ —— $\overset{+}{x_{11}}$

~~12~~th app.

$$x_{12} = \frac{x_{10} + x_{11}}{2} = \frac{2.7401 + 2.7411}{2}$$

$$\boxed{x_{12} = 2.7406}$$

$f(x_{12}) = f(2.7406) = -0.00004 < 0$.

13^{th} app. $\overset{-}{x_{12}}$ —— $\overset{+}{x_{11}}$

$$x_{13} = \frac{x_{12} + x_{11}}{2} = \frac{2.7411 + 2.7406}{2}$$

$$= \frac{5.4817}{2} = 2.74085$$

$x_{13} = 2.7408$.

$f(x_{13}) = 0.00013 > 0.$ $\overset{+}{x_{13}} \overline{\quad\quad} \overset{-}{x_{12}}$

13th app.

$$x_{14} = \frac{x_{13}+x_{12}}{2} = \frac{2.7406 + 2.7408}{2} = \frac{5.4813}{2}$$

$x_{14} = 2.740725.$

$f(x_{14}) = 0.00004 > 0.$ $\overset{+}{x_{14}} \overline{\quad\quad} \overset{-}{x_{12}}$

14th app.

$$x_{15} = \frac{2.7407 + 2.7406}{2}$$

$x_{15} = 2.74065$

$f(x_{15}) = -0.00004 < 0.$ $\overset{+}{x_{14}} \overline{\quad\quad} \overset{-}{x_{15}}$

15th app.

$$x_{16} = \frac{2.7406 + 2.7407}{2}$$

$= 2.74065$

$\boxed{x_{15} = x_{16}}$

2.7406 is root of equation.

(5) By using bisection method, find appropriate root of $\sin x = \frac{1}{x}$.

$f(x) = x\sin x - 1 = 0$.

By trial & error method.

$x=1,\ f(1) = 1\sin 1 - 1 = -0.1585 < 0$

$x=2,\ f(2) = 2\sin 2 - 1 = 0.8185 > 0$

$\therefore$ Root lies b/w 1 & 2.

Let $x_0 = 1,\ x_1 = 2$

$x_0 \qquad\qquad +\ x_1$

First app.

$$x_2 = \frac{x_0+x_1}{2} = \frac{3}{2} = 1.5$$

$x_2 = 1.5$

$f(x_2) = f(1.5) = 0.4962 > 0$

root lies b/w x_0 & x_2.

second app.

$$x_3 = \frac{x_2+x_0}{2} = \frac{1.5+1}{2} = \frac{2.5}{2} = 1.25$$

$x_3 = 1.25$

$f(x_3) = f(1.25) = 0.1862 > 0$

root lies b/w x_3 & x_0.

third app.

$$x_4 = \frac{x_3+x_0}{2} = \frac{1.25+1}{2} = \frac{2.25}{2} = 1.125$$

$x_4 = 1.125$

$f(x_4) = f(1.125) = 0.0150 > 0.$

root lies b/w x_4 & x_0. $\qquad \overset{+}{x_4} \text{———} \overset{-}{x_0}$

fourth app.

$$x_5 = \frac{x_4 + x_0}{2} = \frac{1.125+1}{2} = \frac{2.125}{2} = 1.0625$$

$x_5 = 1.0625.$

$f(x_5) = f(1.0625) = -0.0718 < 0.$

root lies b/w x_5 & x_4 $\qquad \overset{-}{x_5} \text{———} \overset{+}{x_4}$

fifth app.

$$x_6 = \frac{x_5 + x_4}{2} = \frac{1.0625 + 1.125}{2} = 1.0937.$$

$f(x_6) = -0.0284 < 0. \qquad \overset{-}{x_6} \text{———} \overset{+}{x_4}$

root lies b/w x_6 & x_4.

sixth app.

$$x_7 = \frac{x_6 + x_4}{2} = 1.1093.$$

$x_7 = 1.1093.$

$f(x_7) = -0.0067 < 0.$

root lies b/w x_7 & x_4 $\qquad \overset{-}{x_7} \text{———} \overset{+}{x_4}$

seventh app.

$$x_8 = \frac{x_7 + x_4}{2} = 1.1171$$

$x_8 = 1.1171.$

$f(x_8) = 0.0043 > 0 \qquad \overset{+}{x_8} \text{———} \overset{-}{x_7}$

root lies b/w x_8 & x_7.

eight app.

$$x_9 = \frac{x_8 + x_7}{2} = 1.1132 \qquad x_9 = 1.1132$$

$f(x_9) = -0.0013 < 0$.

root lies b/w x_9 & x_8.

9th app:

$$x_{10} = \frac{x_9 + x_8}{2} = 1.1151$$

$x_{10} = 1.1151$.

$f(x_{10}) = 0.0013 > 0$. $\quad \overset{+}{x_{10}} \quad \overset{-}{x_9}$.

10th app.

$$x_{11} = \frac{x_{10} + x_9}{2} = 1.1141$$

$x_{11} = 1.1141$

$f(x_{11}) = -0.00007 < 0$.

root lies b/w x_{11} & x_{10} $\quad \overset{-}{x_{11}} \quad \overset{+}{x_{10}}$

11th app.

$$x_{12} = \frac{x_{11} + x_{10}}{2} = 1.1146.$$

$x_{12} = 1.1146$.

$f(x_{12}) = 0.0006 > 0$.

root lies b/w x_{12} & x_{11}. $\quad \overset{+}{x_{12}} \quad \overset{-}{x_{11}}$.

12th app.

$$x_{13} = \frac{x_{12} + x_{11}}{2} = 1.1143.$$

$\boxed{x_{13} = 1.1143}$

$f(x_{13}) = 0.0001 > 0$ $\quad \frac{+ \quad\quad -}{x_{13} \quad\quad x_{11}}$

root lies b/w x_{13} & x_{11}.

13th app.

$$x_{14} = \frac{x_{13} + x_{11}}{2} = 1.1142.$$

$$x_{14} = 1.1142.$$

$f(x_{14}) = 0.00005 > 0.$ $\quad \frac{+ \quad\quad -}{x_{14} \quad\quad x_{11}}$

root lies b/w x_{14} & x_{11}.

14th app.

$$x_{15} = \frac{x_{14} + x_{11}}{2} = 1.1141$$

$$\boxed{x_{15} = 1.1141}$$

$f(x_{15}) = -0.00007 < 0$ $\quad \frac{- \quad\quad +}{x_{15} \quad\quad x_{14}}$

root lies b/w x_{15} & x_{14}.

15th app.

$$x_{16} = \frac{x_{15} + x_{14}}{2} = 1.1141$$

$$\boxed{x_{16} = 1.1141}$$

Regular - falsi method :- (false position method)

① Let $f(x) = 0$ be the given equation

Find a and b such that $f(a) < 0$ & $f(b) > 0$.

or $f(a) \cdot f(b) < 0$.

Choose $x_0 = a$ & $x_1 = b$.

Root lies b/w x_0 and x_1.

② First approximation is

$$x_2 = \frac{x_0 f(x_1) - x_1 [f(x_0)]}{f(x_1) - f(x_0)}$$

ⓐ If $f(x_2) < 0$ then

second approximation $x_3 = \dfrac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)}$

ⓑ If $f(x_2) > 0$, second approximation is

$$x_3 = \frac{x_0 f(x_2) - x_2 f(x_0)}{f(x_2) - f(x_0)}$$

continue the above procedure until getting accurate value

Problem

① Find root of equation $x\log_{10}x = 1.2$ by using false position method.

sol Given $f(x) = x\log x - 1.2 = 0$.

At $x=1,\ f(1) = -1.2 < 0.$

$x=2,\ f(2) = 2\log 2 - 1.2 = -0.5979 < 0.$

$x=3,\ f(3) = 3\log 3 - 1.2 = 0.2313 > 0.$

root lies b/w 2 & 3.

choose $x_0 = 2$ $x_1 = 3.$

first approximation is $x_2 = \dfrac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} = \dfrac{2f(3) - 3f(2)}{f(3) - f(2)}$

$$x_2 = \frac{2(0.2313) - 3(-0.5979)}{0.2313 + 0.5979}$$

$$\boxed{x_2 = 2.7210}$$

$f(x_2) = 2.7210 \log 2.7210 - 1.2$

$= -0.0171.$

root lies b/w x_2 and x_1.

second app is $x_3 = \dfrac{x_1 \cdot f(x_2) - x_2 \cdot f(x_1)}{f(x_2) - f(x_1)}$

$$x_3 = \frac{3f(2.7210) - (2.7210)f(3)}{f(2.7210) - f(3)} = \frac{3(-0.0171) - (2.7210)(0.2313)}{-0.0171 - 0.2313}$$

$$\boxed{x_3 = 2.7402}$$

$f(x_3) = 2.7402 \log(2.7402) - 1.2$

$= -0.0003 < 0.$

root lies b/w x_3 and x_1

third app $x_4 = \dfrac{x_1(f(x_3)) - x_3 \cdot f(x_1)}{f(x_3) - f(x_1)}$

$$x_4 = \frac{3(-0.0003) - 2.7402(0.2313)}{-0.0003 - 0.2313}$$

$\boxed{x_4 = 2.7405}$.

$f(x_4) = -0.0001 < 0$ $\quad \underset{x_4}{(-)} \longrightarrow \underset{x_1}{(+)}$

Root lies b/w x_4 & x_1

4th approximation is:

$$x_5 = \frac{x_1 f(x_4) - x_4 f(x_1)}{f(x_4) - f(x_1)}$$

$$= \frac{3(-0.0001) - 2.7405(0.2313)}{-0.0001 - 0.2313}$$

$\boxed{x_5 = 2.7406}$

$f(x_5) = -0.00004 < 0$ $\quad \underset{x_5}{(-)} \longrightarrow \underset{x_1}{(+)}$

Root lies b/w x_5 & x_1

5th approximation

$$x_6 = \frac{x_1 f(x_5) - x_5 f(x_1)}{f(x_5) - f(x_1)}$$

$$= \frac{3(-0.00004) - (2.7406)(0.2313)}{-0.00004 - 0.2313}$$

$\boxed{x_6 = 2.7406.}$

Since $x_5 = x_6 = 2.7406$.

2.7406 is root of given functions

$x \log x = 1.2$.

(2) Find the root of the equation $x \cdot e^x = 2$ using false position method.

Given $f(x) = x \cdot e^x - 2 = 0$.

At $x = 1$ $f(1) = +0.7182$.

At $x = 0$ $f(0) = -2$

$$\underset{x_0}{-}\_\_\_\_\_\_\underset{x_1}{+}$$

root lies b/w 0 & 1. $f(x_0) = -2$.

choose $x_0 = 0$ & $x_1 = 1$ $f(x_1) = 0.718$

first app $x_2 = \dfrac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$

$$x_2 = \frac{0(0.7182) - 1.(-2)}{0.7182 + 2} = \frac{2}{2.7182}$$

$\boxed{x_2 = 0.7857}$ $\boxed{x_2 = 0.7357}$

$f(x_2) = -0.4646 < 0$

$$\underset{x_2}{-}\_\_\_\_\_\_\underset{x_1}{+}$$

root lies b/w x_2 & x_1.

second app

$$x_3 = \frac{x_2 \cdot f(x_1) - x_1 f(x_2)}{f(x_1) - f(x_2)}$$

$\boxed{x_3 = 0.8395}$

$$\underset{x_3}{-}\_\_\_\_\_\_\underset{x_1}{+}$$

$f(x_3) = -0.0563$

root lies b/w x_3 & x_1.

third app

$$x_4 = \frac{x_3 \cdot f(x_1) - x_1 f(x_3)}{f(x_1) - f(x_3)}$$

$\boxed{x_4 = 0.8511}$

$f(x_4) = -0.0065 < 0$

$\overset{-}{x_4}$ ——— $\overset{+}{x_1}$

root lies b/w x_4 & x_1

fourth app.

$$x_5 = \frac{x_4 \cdot f(x_1) - x_1 \cdot f(x_4)}{f(x_1) - f(x_4)}$$

$$\boxed{x_5 = 0.8524}$$

$f(x_5) = -0.0008$

x_5 ——— x_1

root lies b/w x_5 & x_1

5th app.

$$x_6 = \frac{x_5 \cdot f(x_1) - x_1 \cdot f(x_5)}{f(x_1) - f(x_5)}$$

$$\boxed{x_6 = 0.8525}$$

$f(x_6) = -0.0004 < 0$

$\overset{-}{x_6}$ ——— $\overset{+}{x_1}$

root lies b/w x_6 & x_1

6th app.

$$x_7 = \frac{x_6 \cdot f(x_1) - x_1 f(x_6)}{f(x_1) - f(x_6)}$$

$$\boxed{x_7 = 0.8525}$$

(1) find the root of the equation $x^3 - 2x - 5 = 0$ using false position method.

Given $f(x) = x^3 - 2x - 5 = 0$.

At $x = 0$ $f(0) = -5 < 0$.

At $x = 1$ $f(1) = 1 - 2 - 5 = -6 < 0$

x_0 At $x = 2$ $f(2) = 8 - 4 - 5 = -1 < 0$.

x_1 At $x = 3$ $f(3) = 27 - 6 - 5 = \boxed{16} > 0$.

root lies b/w 2 & 3.

$$\underset{2\,x_0}{-ve} \quad\text{———}\quad \underset{3\,x_1}{+ve}$$

choose $\boxed{x_0 = 2}$ and $\boxed{x_1 = 3}$.

first approximation

$$x_2 = \frac{x_0 \cdot f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} = 2.0588$$

$$\boxed{x_2 = 2.0588}$$

$$\underset{x_2}{-} \quad\text{———}\quad \underset{x_1}{+}$$

$f(x_2) = -0.3910$.

second approximation.

$$x_3 = \frac{x_2 \cdot f(x_1) - x_1 \cdot f(x_2)}{f(x_1) - f(x_2)}$$

$$\boxed{x_3 = 2.0812}$$

$f(x_3) = -0.1479$

$$\underset{x_3}{-} \quad\text{———}\quad \underset{x_1}{+}$$

third approximation.

root lies b/w x_3 & x_1

$$x_4 = \frac{x_3 \cdot f(x_1) - x_1 f(x_3)}{f(x_1) - f(x_3)}$$

$$\boxed{x_4 = 2.0896}$$

$f(x_4) = -0.0551$.

$$\underset{x_4}{-} \quad\text{———}\quad \underset{x_1}{+}$$

fourth approximation.

$$x_5 = \frac{x_4 \cdot f(x_1) - x_1 f(x_4)}{f(x_1) - f(x_4)}$$

$$\boxed{x_5 = 2.0927}$$

$f(x_5) = -0.0206.$

fifth approximation.

root lies b/w x_5 & x_1

$-$ $+$

x_5 x_1

$$x_6 = \frac{x_5 f(x_1) - x_1 f(x_5)}{f(x_1) - f(x_5)}$$

$$\boxed{x_6 = 2.0938}$$

$-$ $+$

x_6 x_1

$f(x_6) = -0.0083.$

sixth approximation.

root lies b/w x_6 & x_1

$$x_7 = \frac{x_6 \cdot f(x_1) - x_1 f(x_6)}{f(x_1) - f(x_6)}$$

$$\boxed{x_7 = 2.0942}$$

$f(x_7) = -0.0039.$

seventh approximation.

root lies b/w x_7 & x_1

$-$ $+$

x_7 x_1

$$x_8 = \frac{x_7 \cdot f(x_1) - x_1 f(x_7)}{f(x_1) - f(x_7)}$$

$$\boxed{x_8 = 2.0944}$$

$f(x_8) = -0.0016.$

root lies b/w x_8 & x_1

eigth approximation x_9

$-$ $+$

x_1

$$x_9 = \frac{x_8 \cdot f(x_1) - x_1 f(x_8)}{f(x_1) - f(x_8)}$$

$$\boxed{x_9 = 2.0944.}$$

$x_8 = x_9 = 2.0944.$

i.e root is 2.0944.

⑤ Find the root of equation: $x^4 - x - 10 = 0$ using false position method.

Given $f(x) = x^4 - x - 10 = 0$.

At $x = 0$ $\quad f(0) = -10 < 0$

$f(x_0)$ At $x = 1$ $\quad f(1) = 1 - 1 - 10 = -10 < 0$.

$f(x_1)$ At $x = 2$ $\quad f(2) = 16 - 2 - 10 = 4 > 0$

root lies b/w $1 \& 2$

choose $\boxed{x_0 = 1}$ and $\boxed{x_1 = 2}$.

$-$ $\quad$ $+$

x_0 $\quad$ x_1

first approximation.

root lies b/w $x_0 \& x_1$

$$x_2 = \frac{x_0 \cdot f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}.$$

$$\boxed{x_2 = 1.714}$$

$f(x_2) = -3.0833$.

$-$ $\quad$ $+$

x_2 $\quad$ x_1

Second approximation.

root lies b/w $x_2 \& x_1$

$$x_3 = \frac{x_2 f(x_1) - x_1 \cdot f(x_2)}{f(x_1) - f(x_2)}.$$

$$\boxed{x_3 = 1.8384}$$

$f(x_3) = -0.0459$.

third approximation

$-$ $\quad$ $+$

x_3 $\quad$ x_1

root lies b/w $x_3 \& x_1$

$$x_4 = \frac{x_3 \cdot f(x_1) - x_1 f(x_3)}{f(x_1) - f(x_3)}$$

$$\boxed{x_4 = 1.8402.}$$

$f(x_4) = -0.9729$

$-$ $\quad$ $+$

x_4 $\quad$ x_1

4th approximation.

$$x_5 = \frac{x_4 \cdot f(x_1) - x_1 \cdot f(x_4)}{f(x_1) - f(x_4)}$$

$$\boxed{x_5 = 1.8539}$$

$f(x_5) = 0.0413$

$$\underset{x_5}{-}\text{————————}\underset{x_1}{+}$$

fifth approximation.

root lies b/w x_5 & x_1

$$x_6 = \frac{x_5 \cdot f(x_1) - x_1 \cdot f(x_5)}{f(x_1) - f(x_5)}$$

$$\boxed{x_6 = 1.8553.}$$

$f(x_6) = -0.0069$

Sixth approximation.

$$\underset{x_6}{-}\text{————————}\underset{x_1}{+}$$

root lies b/w x_6 & x_1

$$x_7 = \frac{x_6 \cdot f(x_1) - x_1 f(x_6)}{f(x_1) - f(x_6)}$$

$$\boxed{x_7 = 1.8555}.$$

$f(x_7) = -0.0020.$

Seventh approximation.

$$\underset{x_7}{-}\text{————————}\underset{x_1}{+}$$

root lies b/w x_7 & x_1

$$x_8 = \frac{x_7 \cdot f(x_1) - x_1 f(x_7)}{f(x_1) - f(x_7)}$$

$$\boxed{x_8 = 1.85557}$$

(3) $e^x \sin x = 1$

$f(x) = e^x \sin x - 1 = 0.$

at $x=0$, $f(0) = e^0 \sin 0 - 1 = -1 < 0 = f(x_0)$

at $x=1$, $f(1) = e^1 \sin 1 - 1 = 1.2873 > 0 = f(x_1)$

root lies b/w 0 & 1.

choose $x_0 = 0$, $x_1 = 1$.

1st app is $x_2 = \dfrac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$

$= \dfrac{0 f(1) - 1 f(0)}{f(1) - f(0)}$

$\boxed{x_2 = 0.4371.}$

$f(x_2) = -0.3446 < 0.$

root lies b/w x_2 & x_1

2nd app is $x_3 = \dfrac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)}$

$\boxed{x_3 = 0.5559.}$

$f(x_3) = -0.0799 < 0$ $\quad \underset{x_3}{(-)} \text{———} \underset{x_1}{(+)}$

root lies b/w x_3 & x_1

3rd app is $x_4 = \dfrac{x_1 f(x_3) - x_3 f(x_1)}{f(x_3) - f(x_1)}$

$\boxed{x_4 = 0.5818.}$

$f(x_4) = -0.0167 < 0$

root lies b/w x_4 & x_1

4^{th} app is $x_5 = \dfrac{x_1 f(x_4) - x_4 f(x_1)}{f(x_4) - f(x_1)}$

$\boxed{x_5 = 0.5871}$

$f(x_5) = -0.0035 < 0$

root lies b/w x_5 & x_1.

5^{th} app.

$x_6 = \dfrac{x_1 f(x_5) - x_5 f(x_1)}{f(x_5) - f(x_1)}$

$\boxed{x_6 = 0.5882}$

$f(x_6) = -0.0008 < 0$

root lies b/w x_6 & x_1.

6^{th} app

$x_7 = \dfrac{x_1 f(x_6) - x_6 f(x_1)}{f(x_6) - f(x_1)}$

$\boxed{x_7 = 0.5884}$

$f(x_7) = -0.0003 < 0$

7^{th} app

$x_8 = \dfrac{x_1 f(x_8) - x_7 f(x_1)}{f(x_8) - f(x_1)}$ $\boxed{x_8 = 0.5884}$

Iteration method:-

1. set $f(x) = 0$.

2. $x = a$, $x = b$.

$f(a) < 0$
$f(b) > 0$

$x_0 = a$ or $x_0 = b$.

or $x_0 = \frac{a+b}{2}$.

1. let $f(x) = 0$ be the equation.
2. Find a & b such that $f(a) < 0$ & $f(b) > 0$.

or $f(a) f(b) < 0$.

3. choose $x_0 = \frac{a+b}{2}$.
4. Rewrite $f(x) = 0$ as $x = \phi(x)$ $\forall$ $a \leq x \leq b$.

$a \leq \phi(x) \leq b$.

5. If $|\phi'(x)|$ then equation can be used as iteration formula.

i.e $x_i = \phi(x_{i-1})$

Problems

① Find positive root of $x^4 - x - 10 = 0$ by iteration

let $f(x) = x^4 - x - 10 = 0$

$x = 0$ $f(0) = -10 < 0$.

$x = 1$ $f(1) = -10 < 0$

$x = 2$ $f(2) = 4 > 0$.

Root lies b/w 1 & 2.

$x_0 = 1.5 \left(\frac{1+2}{2}\right)$

write $f(x) = 0$ as $x = \phi(x)$

$x^4 - x - 10 = 0$.

$x^4 = x + 10$

$x = (x+10)^{1/4}$.

Here $\phi(x) = (x+10)^{1/4}$.

$\phi'(x) = \frac{1}{4}(x+10)^{-3/4}$.

$\phi'(x) = \frac{1}{4(x+10)^{3/4}}$.

$|\phi'(x)| = \frac{1}{4(x+10)^{3/4}}$.

$|\phi'(x)|_{x=1} = \frac{1}{4(1+10)^{3/4}}$

$= 0.041 < 1$.

$|\phi'(x)|_{x=1} < 1$

$|\phi'(x)|_{x=2} = \frac{1}{4(2+10)^{3/4}}$

$\doteq 0.038 < 1$

$|\phi'(x)|_{x=2} < 1$

$|\phi'(x)| < 1 \ \forall \ 1 \leq x < 2$.

By iteration method

$x_i = \phi(x_{i-1})$

$x_1 = \phi(x_0)$

$= \phi(1.5)$

$= (1.5+10)^{1/4}$

$\boxed{x_1 = 1.8415}$

$x_2 = \phi(x_1)$

$= \phi(1.8415)$

$= (1.8415+10)^{1/4} = 1.8550$.

$x_3 = \phi(x_2)$

$= \phi(1.8550)$

$= (1.8550+10)^{1/4}$.

$\boxed{x_3 = 1.8555}$

$x_4 = \phi(x_3)$

$= \phi(1.8555)$

$= (1.8555+10)^{1/4}$

$\boxed{x_4 = 1.8555}$

$\therefore x_3 = x_4$ are same

Hence 1.8555 is the root of function

$x^4 - x - 10 = 0$.

② Find the positive root of $x^3 - 2x - 5 = 0$ by iteration.

$x=0, f(0) = -5 < 0$.

$x=1, f(1) = -6 < 0$.

$x=2, f(2) = -1 < 0$.

$x=3, f(3) = 16 > 0$.

Root lies b/w 2 & 3.

$x_0 = \frac{2+3}{2} = \frac{5}{2} = 2.5$

write $f(x)=0$ as $x = \phi(x)$

$x^3 - 2x - 5 = 0$.

$x^3 = 2x+5$.

$x = (2x+5)^{1/3}$.

Here, $\phi(x) = (2x+5)^{1/3}$.

$\phi'(x) = \frac{1}{3}(2x+5)^{1/3-1} = \frac{1}{3}(2x+5)^{-2/3}$

$$|\phi'(x)| = \frac{1}{3(2x+5)^{2/3}}$$

$$|\phi'(x)|_{x=2} = \frac{1}{3[2(2)+5]^{2/3}} = \frac{1}{3(4+5)^{2/3}} = \frac{1}{3(9)^{2/3}}$$

$$|\phi'(x)|_{x=2} = \frac{1}{3(4.3267)} = 0.0770 < 2.$$

$$|\phi'(x)|_{x=3} = \frac{1}{3[2(3)+5]^{2/3}} = \frac{1}{3(11)^{2/3}} = \frac{1}{14.8382}.$$

$$|\phi'(x)|_{x=3} = 0.0673 < 2.$$

$$|\phi'(x)| < 2 \quad \forall \; 2 < x < 3.$$

By iteration:

$$x_i = \phi(x_{i-1})$$

$$x_1 = \phi(x_{i-1})$$

$$x_1 = \phi(2.5)$$

$$= (2 \times 2.5 + 5)^{1/3}.$$

$$\boxed{x_1 = 2.1544}$$

$$x_2 = \phi(x_1)$$

$$= \phi(2.1544)$$

$$= (2 \times 2.1544 + 5)^{1/3}.$$

$$\boxed{x_2 = 2.1036.}$$

$$x_3 = \phi(x_2)$$

$$= 2(2.1036)$$

$$= (2 \times 2.1036)^{1/3}$$

$$\boxed{x_3 = 2.0959.}$$

$x_4 = \phi(x_3)$

$= \phi(2.0959)$

$\boxed{x_4 = 2.0947}$

$x_5 = \phi(x_4)$

$x_5 = \phi(2.0947)$

$\boxed{x_5 = 2.0945}$

$x_6 = \phi(x_5)$

$x_6 = \phi(2.0945)$

$\boxed{x_6 = 2.0945}$

③ Evaluate $\sqrt{12}$ by iteration method.

$x = \sqrt{12}$.

$x^2 = 12$.

$f(x) = x^2 - 12 = 0$.

$x \neq 0$

Root lies b/w 3 & 4

$x=0 \quad f(0) = -12 < 0$

$x=1 \quad f(1) = 1-12 = -11 < 0$.

$x=2 \quad f(2) = 4-12 = -8 < 0$

$x=3 \quad f(3) = 9-12 = -3 < 0$.

$x=4 \quad f(4) = 16-12 = 4 > 0$.

choose $x_0 = 3.5$.

$x^2 - 12 = 0$.

$x^2 = 12 \Rightarrow x \cdot x = 12$.

$x = \frac{12}{x} \Rightarrow x = \phi(x)$

where $\phi(x) = \frac{12}{x}$.

$\phi'(x) = \frac{-12}{x^2}$.

$|\phi'(x)| = \frac{12}{x^2}$.

at $x = 4$, $|\phi'(x)| = \frac{12}{16} = 0.75 < 1$.

$x_i = \phi(x_{i-1})$

$x_1 = \phi(x_0)$

$= \phi(3.5)$

$x_1 = \frac{12}{3.5} = 3.4285$.

$x_2 = \phi(3.4285) = \frac{12}{3.4285} = 3.5000$.

$x_3 = \phi(3.5000)$

$= 3.4285$.

④ Find a +ve root of $3x = \cos x + 1$ by iteration

$f(x) = 3x - \cos x - 1 = 0$.

$x = 0$, $f(0) = -2 < 0$.

$x = 1$, $f(1) = 3 - \cos 1 - 1 = 1.4596 > 0$.

root lies b/w 0 & 1

choose $x_0 = \frac{0+1}{2} = 0.5$.

$3x - \cos x - 1 = 0$.

$3x = \cos x + 1$

$\phi(x) = \frac{1}{3}(\cos x + 1)$

$\phi'(x) = \frac{1}{3}(-\sin x)$

$|\phi'(x)| = \frac{1}{3}\sin x$

$|\phi'(x)|$

at $x=0 \Rightarrow \frac{1}{3}(0) = 0<1$.

at $x=1$

$|\phi'(x)| = \frac{1}{3}\sin 1$.

$= 0.2804<1$.

$x_i = \phi(x_{i-1})$

$x_1 = \phi(x_0)$

$= \phi(0.5)$

$= \frac{1}{3}(\cos(0.5)+1)$

$x_1 = 0.6258$.

$x_2 = \phi(x_1)$

$= \phi(0.6258)$

$\boxed{x_2 = 0.6034}$

$x_3 = \phi(x_2) = \phi(0.6034)$

$x_3 = \frac{1}{3}(\cos(0.6034))$

$\boxed{x_3 = 0.6078}$

$x_4 = \frac{1}{3}(\cos(0.6078)+1)$

$= 0.6069$.

$x_5 = \frac{1}{3}(\cos(0.6069)+1)$

$\boxed{x_5 = 0.6071}$

$x_6 = \frac{1}{3}(\cos(0.6071)+1)$

$\boxed{x_6 = 0.6071}$

$\therefore$ The root of given function $3x = \cos x + 1$ is 0.6071 since $x_5 = x_6 = 0.6071$

Newton's raphson method

Let $f(x)=0$.

1. choose initial root x_0.
2. First approximation by newton's raphson is

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Second approximation is

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

Repeat above procedure until getting the accurate root upto 4 decimal places.

1. Using Newton - Raphson method, find the root of equation $f(x) = e^x - 3x$.

$$f(x) = e^x - 3x = 0.$$

$x=0 \quad f(0) = e^0 - 0 = 1$

$x=1 \quad f(1) = e^1 - 3 = -0.2817 < 0.$

Root lies b/w 0 & 1.

choose $x_0 = \frac{1}{2} = 0.5$.

$f(x) = e^x - 3x.$

$f'(x) = e^x - 3.$

first approximaton.

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0.5 - \frac{f(0.5)}{f'(0.5)}$$

$x_1 = 0.5 + 0.1100.$

$\boxed{x_1 = 0.6100.}$

2nd. app.

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 0.6100 - \frac{f(0.6100)}{f'(0.6100)}$$

$$= 0.6100 - \left(\frac{e^{0.6100} - 3(0.6100)}{e^{0.6100} - 3}\right)$$

$\boxed{x_2 = 0.6189}$

3rd approximation.

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= 0.6189 - \frac{f(0.6189)}{f'(0.6189)}$$

$\boxed{x_3 = 0.6190}$

4th approximation

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)}$$

$$= 0.6190 - \frac{f(0.6190)}{f'(0.6190)}$$

$\boxed{x_4 = 0.6190}$

0.6190 is root of $e^x - 3x$.

since $x_3 = x_4 = 0.6190.$

(2) Using newton - Raphson method find the root of equation $f(x) = x \sin x + \cos x = 0$.

$$f(x) = x \sin x + \cos x = 0$$

$x = 0 \quad f(0) = 1 > 0$,

$x = 1 \quad f(1) = 1.3817 > 0$.

$x = 2 \quad f(2) = 1.4024 > 0$.

$x = 3 \quad f(3) = -0.5666 < 0$.

Root lies b/w 2 & 3.

choose $x_0 = \frac{5}{2} = 2.5$.

$$f(x) = x \sin x + \cos x = 0.$$

$$f'(x) = x.\cos x + \sin x - \sin x$$

first approximation -

choose $x_0 = 2.5$.

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 2.5 + \frac{0.26950}{0.}$$

$$= 2.5 - \frac{2.5 \sin(2.5) + \cos(2.5)}{2.5 \cos(2.5) - \sin(2.5)}$$

$x_1 = 2.7671$.

2nd app:

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$x_2 = 2.7671 - \frac{0.0815}{-2.9411}$$

$$x_2 = 2.7671 + 0.0277.$$

$$\boxed{x_2 = 2.7948}$$

3rd app.

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= 2.7948 + 0.0031$$

$$\boxed{x_3 = 2.7979}.$$

4th app:

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)}$$

$$= 2.7979 + 0.0004$$

$$\boxed{x_4 = 2.7983}$$

5th app:

$$x_5 = x_4 - \frac{f(x_4)}{f'(x_4)}$$

$$= 2.7983 + 0.00006.$$

$$x_5 = 2.79836.$$

$\therefore$ 2.7983 is root of $x\sin x + \cos x = 0.$

since $x_4 = x_5 = 2.7983$

(5) Find root of $e^x \sin x = 1$ using NRM

$f(x) = e^x \sin x - 1 = 0$.

$x = 0 \Rightarrow f(0) = -1 < 0$

$x = 1 \Rightarrow f(1) = 1.28733 > 0$

root lies b/w 0 & 1.

$x_0 = \frac{0+1}{2} = 0.5$

$f'(x) = e^x \sin x - 1$

$= e^x \sin x + e^x \cos x$.

1st approx.

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0.5 - \frac{e^{(0.5)} \sin(0.5) - 1}{e^{(0.5)} \sin(0.5) + e^{(0.5)} \cos(0.5)}$

$= 0.5936$.

2nd approx.

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0.5885$

3rd approx

$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 0.5885$

$\therefore$ 0.5885 is root of $e^x \sin x - 1 = 0$.

⑥ Using Newton Rhapson method find ⓐ square root of a number. ⓑ Find reciprocal of a number.

Sol

ⓐ N be the number whose square root is to be found

$x = \sqrt{N}$.

$x^2 - N = 0$. (or) $f(x_i) = x_i^2 - N$.

By Newton's Rhapson method

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

$$f'(x_i) = 2x_i.$$

$$x_{i+1} = x_i - \left[\frac{x_i^2 - N}{2x_i}\right]$$

$$= \frac{2x_i^2 - x_i^2 + N}{2x_i}.$$

$$= \frac{x_i^2 + N}{2x_i}$$

$$\boxed{x_{i+1} = \frac{x_i}{2} + \frac{N}{2x_i}.}$$

ⓑ $x = \frac{1}{N}$.

$\frac{1}{x} = N$.

$f(x) = \frac{1}{x} - N = 0$.

$f(x_i) = \frac{1}{x_i} - N = 0$.

$f'(x_i) = \frac{-1}{x_i^2}$.

By Newton - Raphson method.

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

$$= x_i - \frac{\frac{1}{x_i} - N}{\frac{-1}{x_i^2}}$$

$$= x_i + x_i^2\left(\frac{1}{x_i} - N\right)$$

$$= x_i + \frac{x_i^{\cancel{2}}}{\cancel{x_i}} - N x_i^2$$

$$= 2x_i - N x_i^2.$$

$$\boxed{x_{i+1} = x_i\,[2 - N x_i]}$$

(b) Using Newton's Rhapson method.

① Find square root of 10.

② Find a reciprocal of 22.

Sol ① square root of 10.

$f(x) = x^2 - 10 = 0.$ Here $N = 10.$

Root lies b/w 3 & 4.

$x_0 = 3$

By newton's rhapson method:

$$x_{i+1} = \frac{x_i}{2} + \frac{N}{2x_i} = \frac{1}{2}\left[x_i + \frac{N}{x_i}\right]$$

First approximation.

$$x_1 = \frac{1}{2}\left(3 + \frac{10}{3}\right) = 3.1666.$$

Second approximation.

$$x_2 = \frac{1}{2}\left[3.1666 + \frac{10}{3.1666}\right] = \underline{3.1622.}$$

third approximation.

$$x_3 = \frac{1}{2}\left[3.1622 + \frac{10}{3.1622}\right] = \underline{3.1622.}$$

Hence square root of $10 = 3.1622.$

② Reciprocal:-

$x = \frac{1}{29}$..

or $f(x) = \frac{1}{x} - 22 = 0$.

0.04545

Choose $x_0 = 0.04$.

By newton raphson method.

$x_{i+1} = x_i [2 - N x_i]$

$x_1 = x_0 [2 - N x_0]$

$= 0.04 [2 - 22 (0.04)]$

$= 0.04 [2 - 0.88]$

$\boxed{x_1 = 0.0448}$

$x_2 = x_1 [2 - N x_1]$

$= 0.0448 [2 - 22 (0.0448)]$

$\boxed{x_2 = 0.04544}$

$x_3 = x_2 [2 - N x_2]$

$= 0.0454 [2 - N (0.0454)]$

$\boxed{x_3 = 0.0454}$

0.0454 is the value of reciprocal of 22.

(4) Using newton's rhapson method.

(1) Using find a root of 24.

(2) find reciprocal of 18.

(1) square root of 24.

$f(x) = x^2 - 24 = 0.$

$N = 24.$

root lies b/w 4 & 5.

$x_0 = 5.$

By newton's rhapson method.

$x_{i+1} = \frac{1}{2}\left(x_i + \frac{N}{x_i}\right)$

$x_1 = \frac{1}{2}\left(x_0 + \frac{N}{x_0}\right) = \frac{1}{2}\left[5 + \frac{24}{5}\right]$

$\boxed{x_1 = 4.9}$

$x_2 = \frac{1}{2}\left(x_1 + \frac{N}{x_1}\right] = \frac{1}{2}\left[4.9 + \frac{24}{4.9}\right]$

$\boxed{x_2 = 4.8989}$

$x_3 = \frac{1}{2}\left[x_2 + \frac{N}{x_2}\right] = \frac{1}{2}\left[4.8989 + \frac{24}{4.8989}\right]$

$\boxed{x_3 = 4.8989}$

Hence square root of $24 = 4.8989.$

② Reciprocal

$x = \frac{1}{18}$.

$f(x) = \frac{1}{x} - 18 = 0$.

choose $x_0 = 0.05$.

By newton Rhapson method:

$x_{i+1} = x_i[2 - N \cdot x_i]$.

$x_{0+1} = x_0[2 - N x_0]$

$x_1 = 0.05\,[2 - 18(0.05)]$

$= 0.055$

$x_2 = 0.055\,(2 - 18 \times 0.055)$

$x_2 = 0.0555$

$x_3 = 0.0555\,(2 - 18 \times 0.0555)$

$x_3 = 0.0555$

0.0555 is reciprocal of 18.

Numerical integration:-

1) Trapezoidal rule.

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + y_3 - - -)\right]$$

Here $h = \frac{b-a}{n}$.

$x_1 = x_0 + h$ $\quad x_2 = x_1 + h$ or $x_0 + 2h$ $\quad x_3 = x_2 + h$ $\quad x_0 + 3h$.

2) Simpson's $1/3^{rd}$ rule:-

$$\int_a^b f(x)\,dx = \frac{h}{3}\left[(y_0 + y_n) + 2(y_2 + y_4 + y_6 - - -) + 4(y_1 + y_3 + y_5 + - - -)\right]$$

3) Simpson's $3/8^{th}$ rule:-

$$\int_a^b f(x)\,dx = \frac{3h}{8}\left[(y_0 + y_n) + 2(y_3 + y_6 + y_9 + - - -) + 4(y_1 + y_2 + y_4 + y_5 + y_7 - - -)\right]$$

Note:-

Simpson's $1/3^{rd}$ rule is used only when n is multiple of 2.

Simpson's $3/8^{th}$ rule is used only when n is multiple of 3.

Problems:

① Evaluate $\int_0^1 x^3 dx$ with five sub intervals by trapezoidal rule.

given $n = 5$, $b = 1$, $a = 0$.

$h = \frac{b-a}{n} = \frac{1-0}{5} = \frac{1}{5}$.

$x_0 = 0$ $\qquad y_0 = x_0^3 = 0$.

$x_1 = x_0 + h$ $\qquad y_1 = x_1^3 = \frac{1}{(5)^3} = \frac{1}{125} = 0.008$

$= 0 + 1/5$

$x_2 = x_1 + h$ $\qquad y_2 = x_2^3 = \left(\frac{2}{5}\right)^3 = 0.064$.

$= \frac{1}{5} + \frac{1}{5}$

$= 2/5$,

$x_3 = x_2 + h$ $\qquad y_3 = (3/5)^2 = 0.216$

$= 3/5$

$x_4 = x_3 + h$ $\qquad y_4 = (4/5)^3 = 0.512$.

$= 4/5$.

$x_5 = x_4 + h$. $\qquad y_4 = (5/5)^3 = 1$.

$= 1$

Trapezoidal rule:

$\int_0^1 x^3 dx = \frac{h}{2}\left[(y_0 + y_5) + 2(y_1 + y_2 + y_3 + \cdots)\right]$

$= \frac{1}{10}\left[1 + 2(0.008 + 0.064 + 0.216 + 0.512)\right]$

$= 0.26$.

② Evaluate $\int_a^b \frac{1}{1+x}\,dx$ using ① simpson's 1/3 rule.
② simpsons 3/8 rule

→ compare result with initial value.

simpson's 1/3 rule.

$$\int_a^b = f(x)\,dx = \frac{h}{3}\left((y_0+y_n) + 2(y_2+y_4+y_6+\cdots) + 4(y_1+y_3+y_5+\cdots)\right)$$

$$h = \frac{b-a}{n} = \frac{6-0}{6} = 1$$

$x_a = a = 0 \qquad x_n = b = 6.$

$x_0 = 0 \qquad y_0 = \frac{1}{1+x_0}.$

$y_0 = 1$

$y_1 = 0.5.$

$x_1 = 1$

$x_2 = x_1 + h.$
$= 1+1$
$= 2$

$y_2 = 1/3 = 0.333.$

$y_3 = 1/4 = 0.25.$

$x_3 = x_2 + h$
$= 3.$

$x_4 = x_3 + h.$
$= 4.$

$y_4 = 1/5 = 0.2.$

$x_5 = x_4 + h$
$= 5.$

$y_5 = 1/6 = 0.1666$

$x_6 = x_5 + h.$
$= 6$

$y_6 = \frac{1}{7} = 0.1428.$

Simpson's 1/3 rule.

$$\int_0^6 \frac{1}{1+x}\,dx = \frac{h}{3}\left[(y_0+y_6) + 2(y_2+y_4) + 4(y_1+y_3+y_5)\right]$$

$$= \frac{1}{3}\left(1.1428 + 2(0.5333) + 4(0.5+0.25+0.1666)\right)$$

$$= 1.9586.$$

② Simpson's 3/8th rule.

$$\int_0^6 \frac{1}{1+x}\,dx = \frac{3h}{8}\left((y_0+y_6) + 2(y_3) + 3(y_1+y_2+y_4+y_5 \sim\right)$$

$$= \frac{3}{8}\left(1.1428 + 2(0.25) + 3(0.5+0.3333+0.2 + 0.1666)\right)$$

$$= 5.24178 \times \frac{3}{8}$$

$$= 1.965.$$

③ $h = \frac{b-a}{n} \Rightarrow n = \frac{1}{0.1}$

$n = 10.$

$\int_0^1 \sqrt{1+x^3}\,dx$ $h = 0.1$ using.

① 1/3rd rule ② Trapezoidal rule.

Simple 1/3 rule.

$$\int_0^1 \sqrt{1+x^3}\,dx = \frac{h}{3}\left[(y_0+y_{10}) + 2(y_2+y_4+y_6+y_8) + 4(y_1+y_3+y_5+y_7+y_9)\right]$$

$x_0 = 0$ $y_0 = 1$

$x_1 = x_0 + h.$ $y_1 = \sqrt{1+(0.1)^3}$

$= 0.1$ $= 1.0004$

$x_2 = x_1 + h.$ $\qquad y_2 = \sqrt{1+(0.2)^3}$

$= 0.2$ $\qquad = 1.0039.$

$x_3 = 0.3.$ $\qquad y_3 = 1.0134.$

$x_4 = 0.4$ $\qquad y_4 = 1.0315.$

$x_5 = 0.5$ $\qquad y_5 = 1.0606.$

$x_6 = 0.6$ $\qquad y_6 = 1.1027.$

$x_7 = 0.7$ $\qquad y_7 = 1.1588.$

$x_8 = 0.8$ $\qquad y_8 = 1.2296.$

$x_9 = 0.9$ $\qquad y_9 = 1.31491$

$x_{10} = 1$ $\qquad y_{10} = 1.4142.$

$$① \Rightarrow \frac{0.1}{3}\Big[(1 + 1.4142) + 2(1.0039 + 1.0315 + 1.1027 + 1.2296)$$

$$+ 4(1.0004 + 1.0134 + 1.0606 + 1.1588 + 1.3149)\Big]$$

$$= 1.1114.$$

$$② \Rightarrow \int_a^b \sqrt{1+x^3}\, dx = \frac{h}{2}\Big[(y_0 + y_{10}) + 2(y_1 + y_2 + y_3 + y_4 \cdots + y_9)\Big]$$

$$= 1.1122.$$

④ Evaluate $\int_0^1 \frac{dx}{1+x^2}$ using Simpson $\frac{3}{8}$ rule $h=\frac{1}{6}$

$h = \frac{b-a}{n} \Rightarrow \frac{1}{6} = \frac{1-0}{n} \Rightarrow n=6.$

$x_0 = 0 \qquad y_0 = 1$

$x_1 = x_0 + h. \qquad y_1 = \frac{1}{1+(\frac{1}{6})^2} = \frac{1}{1+0.0277}.$

$= \frac{1}{6}. \qquad y_1 = 0.9730.$

$x_2 = x_1 + h. \qquad y_2 = \frac{1}{1+(\frac{1}{3})^2} = \frac{1}{1+0.1111}$

$= \frac{2}{6} = \frac{1}{3}. \qquad y_2 = 0.9000.$

$x_3 = x_2 + h. \qquad y_3 = \frac{1}{1+(\frac{1}{2})^2} = \frac{1}{1+0.25}$

$= \frac{3}{2} = \frac{1}{2}. \qquad y_3 = 0.8$

$x_4 = x_3 + h. \qquad y_4 = \frac{1}{1+(\frac{2}{3})^2} = \frac{1}{1+0.4444}$

$= \frac{4}{6} = \frac{2}{3}. \qquad y_4 = 0.6923.$

$x_5 = x_4 + h. \qquad y_5 = \frac{1}{1+(\frac{5}{6})^2} = \frac{1}{1+0.4444}.$

$= \frac{5}{6}. \qquad y_5 = 0.5901.$

$x_6 = x_5 + h. \qquad y_6 = \frac{1}{1+(1)^2} = \frac{1}{1+1}$

$= 1 \qquad y_6 = 0.5.$

Simpson's $\frac{3}{8}$th rule.

$$\int_a^b f(x)dx = \frac{3h}{8}\left[(y_0+y_n)+2(y_3+y_6+y_9+\cdots)+3(y_1+y_2+y_4\cdots)\right]$$

$$= \frac{3\times\frac{1}{6}}{8}\left[(1+0.5)+2(0.8)+3(0.9729+0.9+0.6923+0.5901)\right]$$

$= 0.7853.$

(5) Dividing the range into 10 equal parts, find an approximate value of $\int_0^{\pi}\sin x\,dx$.

by (1) Trapezoidal rule.

$h = \frac{b-a}{n} = \frac{\pi-0}{10} = \frac{\pi}{10}.$

$x_0 = 0$ $\qquad$ $y_0 = 0.$

$x_1 = x_0 + h$ $\qquad$ $y_1 = \sin x_1$

$= 0 + \frac{\pi}{10}$ $\qquad$ $= \sin\frac{\pi}{10}$

$= \frac{\pi}{10}$ $\qquad$ $y_1 = 0.3090.$

$x_2 = x_1 + h$ $\qquad$ $y_2 = \sin x_2.$

$= \frac{\pi}{10}+\frac{\pi}{10}$ $\qquad$ $= \sin\frac{\pi}{5}$

$= \frac{\pi}{5}$ $\qquad$ $= 0.5877$

$x_3 = x_2 + h.$ $\qquad$ $y_3 = \sin x_3$

$= \frac{3\pi}{10}$ $\qquad$ $= \sin\frac{3\pi}{10} = 0.8090.$

$x_4 = \frac{4\pi}{10}$ $\qquad$ $y_4 = \sin\frac{4\pi}{10}$

$y_4 = 0.9510.$

$x_5 = \frac{5\pi}{10}$ $\qquad$ $y_5 = \sin\frac{5\pi}{10}$

$y_5 = 1.$

$x_6 = \frac{6\pi}{10}$ $\qquad$ $y_6 = 0.9510.$

$x_7 = \frac{7\pi}{10}$ $\qquad$ $y_7 = \sin\frac{7\pi}{10}$.

$y_7 = 0.8090$.

$x_8 = \frac{8\pi}{10}$ $\qquad$ $y_8 = \sin\frac{8\pi}{10} = 0.5877$.

$x_9 = \frac{9\pi}{10}$ $\qquad$ $y_9 = \sin\frac{9\pi}{10} = 0.3090$.

$x_{10} = \frac{10\pi}{10}$ $\qquad$ $y_{10} = 0$.

① Trapezoid rule.

$$\int_0^\pi \sin x\,dx = \frac{h}{2}\left[(y_0+y_{10}) + 2[y_1+y_2+y_3+y_4 \cdots + y_9]\right]$$

$$= \frac{\frac{\pi}{10}}{2}\left[(0+0) + 2(0.3090+0.5877+0.8090+0.9510+1+0.9510+0.8090+0.5877+0.3090)\right]$$

$$\int_0^\pi \sin x\,dx = 1.9843.$$

② Simpson's rule.

$$\int_0^\pi \sin x\,du = \frac{h}{3}\left[(y_0+y_{10}) + 2(y_2+y_4+y_6+y_8) + 4(y_1+y_3+y_5+y_7+y_9)\right]$$

$$= 2.00002.$$

Numerical solution of ordinary D.E.

Taylor's Series:

To find solution of ordinary DE $\frac{dy}{dx} = f(x,y)$ — ①

given initial condition are $y(x_0) = y_0$ then.

$$y_1 = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \cdots$$

Here $h = x - x_0$.

To find ~~$x_1 = x_0$~~ y_2 at x_2.

$$y_2 = y_1 + \frac{h}{1!} y_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + \cdots$$

Problem:- ① Using Taylor's method, solve $\frac{dy}{dx} = x^2 + y^2$

for $x = 0.4$, given that $y = 0$ at $x = 0$

Given $\frac{dy}{dx} = f(x,y)$

$f(x,y) = x^2 + y^2$

$y' = x^2 + y^2$

$y_0 = 0$, $x_0 = 0$, Take $h = 0.4$.

$y_0' = x^2 + y^2$

$y_0' = 0$

$y'' = 2x + 2yy'$

$y_0'' = 2x_0 + 2y_0 y_0'$

$= 0$

$y''' = 2 + 2[yy'' + (y')^2]$

$y_0''' = 2 + 2[y_0 y_0'' + (y_0')^2]$

$y_0''' = 2$

$y_0^{IV} = 0 + 2[yy''' + y'y'' + 2(y')(y'')]$

$y_0^{IV} = 2[y_0 y_0''' + y_0' y_0'' + 2y_0' y_0'']$

$= 2[0 + 0 + 0]$

$y_0^{IV} = 0$

By taylor's series method.

$y_1 = y_0 + \frac{h}{1!} + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \cdots$

$= 0 + (0.4)(0) + \frac{(0.2)^2}{2}(0) + \frac{(0.4)^3}{6} 2 + \cdots$

$y_1 = \frac{(0.4)^3}{3} = 0.02133.$

② Solve $y' = x - y^2$, $y(0) = 1$ using Taylor's series method and compute $y(0.1)$, $y(0.2)$.

Given $f(x, y) = x - y^2$.

$y' = x - y^2$

Initial condition is $y(0) = 1$.

$x_0 = 0$, $y_0 = 1$

$y' = x - y^2$

$y'' = 1 - 2yy'$

$y''' = 0 - 2[yy'' + y'y']$

$y''' = -2[yy'' + (y')^2]$

$y_0' = x_0 - y_0^2$

$y_0' = 0 - (1)^2$

$= -1$

$y_0'' = 1 - 2 y_0 y_0'$

$= 1 - 2(1)(-1)$

$y_0'' = 3$

$y_0''' = -2[y_0 y_0'' + (y_0')^2]$

$= -2[(1)(3) + (-1)^2]$

$= -2[3+1] = -8.$

By taylor's series,

y_1 at $x_1 = x_0 + h = 0 + 0.1$

$= 0.1$

$y_1 = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \cdots$

$= 1 + (0.1)(-1) + \frac{(0.1)^2}{2}(3) + \frac{(0.1)^3}{6}(-8)$

$= 0.9137.$

$y_1 = 0.9137$ at $x_1 = 0.1$

To find y_2 at $x_2 = x_1 + h$.

$= 0.1 + 0.1$

$= 0.2$.

$y_1' = x_1 - y_1^2$

$= 0.1 - (0.9137)^2$

$= -0.7348$

$y_1'' = 1 - 2y_1 y_1'$

$= 1 - 2(0.9137)(-0.7348)$

$= 2.3427$.

$y_1''' = -2[y_1 y_1'' + (y_1')^2]$

$= -2[(0.9137)(2.3427) + (-0.7348)^2]$

$= -2[2.1405 + 0.5399]$

$= -2[2.6804]$

$= -5.3609$.

$y_2(0.2) = y_1 + \frac{h}{1!} y' + \frac{h^2}{2!} y'' + \frac{h^3}{3!} y_1''' + \cdots$

$= 0.9137 + 0.1(-0.7348) + \frac{(0.1)^2}{2}(2.3427) +$

$\frac{(0.1)^3}{3!}(-5.3609)$

$y_2 = 0.85104$.

(3) Solve $y' - 2y = 3e^x$, $y(0) = 0$ using Taylor's series and compute $y(0.2)$.

Given $f(x,y) = 2y + 3e^x$.

$y' = 2y + 3e^x$.

Initial condition is $y(0) = 0$.

$x_0 = 0$, $y_0 = 0$.

$y' = 2y + 3e^{x}$.

$y'' = 2y' + 3e^{x}$.

$y''' = 2y'' + 3e^{x}$.

$y_0' = 2y_0 + 3e^{x_0}$
$= 2(0) + 3e^{0}$
$= 3$.

$y_0'' = 2y_0' + 3e^{x_0}$
$= 2(3) + 3e^{0}$
$= 9$.

$y_0''' = 2y'' + 3e^{x_0}$
$= 2(9) + 3e^{0}$.
$= 21$.

By Taylor's series

y_1 at $x_1 = x_0 + h$.
$= 0 + 0.2 = 0.2$.

$$y_1 = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \cdots$$

$$= 0 + \frac{0.2}{1}(3) + \frac{(0.2)^2}{2}(9) + \frac{(0.2)^3}{6}(21)$$

$= 0.6 + 0.18 + 0.028$.

$y_1(0.2) = 0.808$.

4. Tabulate $y(0.1), y(0.2)$ & $y(0.3)$ using Taylor's series method given that $y' = y^2 + x$ and $y(0) = 1$

Given $f(x,y) = y^2 + x$.

$y' = y^2 + x$.

Initial condition is $y(0) = 1$

$x_0 = 0,\ y_0 = 1,\ h = 0.1$

$y' = y^2 + x$.

$y'' = 2yy' + 1$

$y''' = 2[yy'' + (y')^2]$

$y_0' = y_0^2 + x_0$

$y_0' = 1$.

$y_0'' = 2y_0y_0' + 1$
$= 2(1)(1) + 1$

$y_0'' = 3$.

$y_0''' = 2[y_0 y_0'' + (y_0')^2]$

$= 2[(1)(3) + (1)^2]$

$y_0''' = 8.$

By Taylor's series.

y_1 at $x_1 = x_0 + h.$

$= 0 + 0.1$

$x_1 = 0.1$

$y_1(0.1) \Rightarrow y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \cdots$

$= (1) + \frac{0.1}{1}(1) + \frac{(0.1)^2}{2}(3) + \frac{(0.1)^3}{6}(8)$

$y_1(0.1) = 1.1163.$

$y_1 = 1.1163$ at $x_1 = 0.1$

To find y_2 at $x_2 = x_1 + h. = 0.1 + 0.1$

$= 0.2.$

$y_1' = y_1^2 + x_1$

$= (1.1163)^2 + 0.1$

$y_1' = 1.3461$

$y_1'' = 2 y_1 y_1' + 1.$

$= 2(1.1163)(1.3461) + 1$

$y_1'' = 4.0053.$

$y_1''' = 2[y_1 y_1'' + (y_1')^2]$

$= 12.5662.$

$y_2 = y_1 + \frac{h}{1!} y_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + \cdots$

$= 1.1163 + (0.1)(1.3461) + \frac{(0.1)^2}{2}(4.0053) + \frac{(0.1)^3}{6}(12.5662)$

$\boxed{y_2 = 1.2730}$ at $x_2 = x_1 + h = 0.2$

To find y_3 at $x_3 = x_2 + h = 0.2 + 0.1$
$= 0.3.$

$y_2' = y_2^2 + x_2.$
$= (1.2730)^2 + (0.2)$

$y_2' = 1.8205.$

$y_2'' = 2y_2 y_2' + 1 = 2(1.2730)(1.8205) + 1$

$y_2'' = 5.6349.$

$y_2''' = 2\left[y_2 y_2'' + (y_2')^2\right] = 2\left[(1.2730)(5.6349) + (1.8205)^2\right]$

$= 20.9748.$

$y_3 = 1.2730 + (0.1)(1.8205) + \frac{(0.1)^2}{2}(5.6349) + \frac{(0.1)^3}{6}(20.9748)$

$y_3 = 1.4867.$

(b) Solve $y' = x+y$ given $y(1) = 0$, find $y(1.1)$ $y(1.2)$ by Taylor's series method.

Given $f(x,y) = x+y.$
$y' = x+y.$

$y(1) = 0$

$x_0 = 1, \; y_0 = 0, \; h = 0.1$

$y' = x+y.$
$y'' = 1+y'.$
$y''' = 0 + y''$
$y^{iv} = y'''$

$y_0' = x_0 + y_0$
$y_0' = 1+0 = 1$
$y_0'' = 1 + y_0' = 1+1 = 2$
$y_0''' = 0 + y_0'' = 0+2 = 2$
$y_0^{iv} = 2.$

By Taylor's series

y_1 at $x_1 = x_0 + h$
$= 1 + 0.1$
$= 1.1$

$$y_1(1.1) = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \frac{h^4}{4!} y_0^{iv}$$

$$= 0 + \frac{0.1}{1}(1) + \frac{(0.1)^2}{2}(2) + \frac{(0.1)^3}{6}(2) + \frac{(0.1)^4}{24}(2)$$

$y_1(1.1) = 0.1103$

$y_1 = 0.1103$ at $x_1 = 1.1$

To find y_2 at $x_2 = x_1 + h$
$= 1.1 + 0.1$
$= 1.2$

$y_1' = x_1 + y_1 = 1.1 + 0.1103 = 1.2103$

$y_1'' = 1 + y_1' = 1 + 1.2103 = 2.2103$

$y_1''' = y_1'' = 2.2103$

$y_1^{iv} = 2$

$$y_2 = y_1 + \frac{h}{1!} y_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + \frac{h^4}{4!} y_1^{iv}$$

$$= 0.1103 + \frac{0.1}{1}(1.2103) + \frac{(0.1)^2}{2}(2.2103) + \frac{(0.1)^3}{6}(2.2103) + \frac{(0.1)^4}{24}(2)$$

$y_2 = 0.2427$

Euler's method :-

To find the solution of ordinary D.E $\frac{dy}{dx} = f(x,y)$ subject to the condition $y(x_0) = y_0$ is given by

y_1 at $(x_1 = x_0 + h) = y_0 + h f(x_0, y_0)$

To find $y_2(x_2) = y_1 + h f(x_1, y_1)$

In general.

$$\boxed{y_{n+1} = y_n + h f(x_n, y_n)}$$

Problems

① Solve by euler's method $y' = x+y$, $y(0) = 1$ and find $y(0.3)$ taking step size $h = 0.1$.

Given

$f(x,y) = x+y$, $x_0 = 0$ $y_0 = 1$. $h = 0.1$

By euler's method.

To find y_1 at $x_1 = x_0 + h = 0 + 0.1 = 0.1$

$$\begin{aligned} y_1(0.1) &= y_0 + h f(x_0, y_0) \\ &= 1 + 0.1 f(0,1) \\ &= 1 + 0.1(0+1) = 1 + 0.1 \quad [\because f(x,y) = x+y] \\ &= 1.1 \end{aligned}$$

To find y_2 at $x_2 = x_1 + h = 0.1 + 0.1$
$= 0.2$.

$$\begin{aligned} y_2(0.2) &= y_1 + h f(x_1, y_1) \\ &= 1.1 + 0.1 f(0.1, 1.1) \\ &= 1.1 + 0.1(0.1 + 1.1) \\ &= 1.1 + 0.1[1.2] \\ y_2 &= 1.22 \text{ at } x_1 = 0.2. \end{aligned}$$

To find y_3 at $x_3 = x_2 + h$

$= 0.2 + 0.1$

$= 0.3$

$y_3(0.3) = y_2 + hf(x_2, y_2)$

$= 1.22 + 0.1 f(0.2, 1.22)$

$= 1.22 + 0.1(1.42)$

$\boxed{y_3(0.3) = 1.362}$

② Using euler's method, solve for y at $x=2$ from $\frac{dy}{dx} = 3x^2+1$, $y(1) = 2$ taking step size

① $h = 0.5$ ② $h = 0.25$ $x_0 = 1$ $y_0 = 2$.

To find y_1 at $x_1 = x_0 + h$

$= 1 + 0.5$

$x_1 = 1.5$

$y_1(1.5) = y_0 + hf(x_0, y_0)$

$= 2 + (0.5)(3(1)^2 + 1)$

$= 2 + (0.5)(4)$

$\boxed{y_1(1.5) = 4}$

To find y_2 at $x_2 = x_1 + h$

$= 1.5 + 0.5$

$= 2$

$y_2(2) = y_1 + hf(x_1, y_1)$

$= 4 + (0.5)(3(1.5)^2 + 1)$

$= 4 + (0.5)(3(2.25) + 1)$

$= 4 + (0.5)(6.75 + 1) = 4 + \frac{1}{2}(\cancel{7.75})$ 3.8725

$= \cancel{7}.87\cancel{0}5$

(2) $h = 0.25$.

To find y_1 at $x_1 = x_0 + h$.
$= 1 + 0.25$.
$x_1 = 1.25$.

$y_1(1.25) = y_0 + hf(x_0, y_0)$
$= 2 + (0.25)(3(1)^2 + 1) = 2 + (0.25)(4)$

$\boxed{y_1(1.25) = 3}$

To find y_2 at $x_2 = x_1 + h$.
$= 1.25 + 0.25 = 1.5$.

$y_2(1.5) = y_1 + hf(x_1, y_1)$
$= 3 + (0.25)(3(1.25)^2 + 1)$
$= 3 + (0.25)(5.6875)$

$\boxed{y_2 = 4.421875}$

To find y_3 at $x_3 = x_2 + h$.
$= 1.5 + 0.25$.
$= 1.75$.

$y_3(1.75) = y_2 + hf(x_2, y_2)$
$= 4.421875 + (0.25)[3(1.5)^2 + 1]$
$= 4.421875 + (0.25)(7.75)$

$\boxed{y_3 = 6.3593}$

To find y_4 at $x_4 = x_3 + h$.
$= 1.75 + 0.25$.
$= 2$.

$y_4(2) = y_3 + hf(x_3, y_3)$
$= 6.3593 + (0.25)(3(1.75)^2 + 1)$

$\boxed{y_4(2) = 8.9061}$

③ Solve numerically using euler's method

$y' = y^2 + x$, $y(0) = 1$, find $y(0.1)$ & $y(0.2)$

let $\boxed{h = 0.1}$ $x_0 = 0$ $y_0 = 1$.

To find y_1 at $x_1 = x_0 + h$.

$= 0 + 0.1$

$= 0.1$.

$y_1(0.1) = y_0 + h.f(x_0, y_0)$

$= 1 + (0.1)(1^2 + 0)$

$= 1 + (0.1)(1)$

$\boxed{y_1(0.1) = 1.1}$

To find y_2 at $x_2 = x_1 + h$.

$= 0.1 + 0.1$

$= 0.2$.

$y_2(0.2) = y_1 + hf(x_1, y_1)$

$= 1.1 + (0.1).(1.21 + 0.1)$

$= 1.1 + (0.1)(1.31)$

$= 1.1 + 0.131$

$\boxed{y_2(0.2) = 1.231}$

Modified euler's method.

To find the solution of ordinary D.E $\frac{dy}{dx} = f(x,y)$, give $y(x_0) = y_0$ is given by.

$$y_1^{(0)} = y_0 + hf(x_0, y_0)$$

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$$

and so on.

Repeat above procedure untill two successive values are similar upto 4 decimal places.

To find y_2 at x_2.

$$y_2^{(0)} = y_1 + hf(x_1, y_1)$$

$$y_2^{(1)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(0)})\right]$$

$$y_2^{(2)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(1)})\right]$$

— — — —

Repeat the above procedure untill two successive values are similar upto 4 decimal places.

① Using modified euler's method find $y(0.2)$ and $y(0.4)$ given $y' = y + e^x$, $y(0) = 0$.

Sol. Given $f(x,y) = y + e^x$, $x_0 = 0, y_0 = 0, h = 0.2$.

By modified euler's method:

$$y_1^{(0)} = y_0 + h f(x_0, y_0)$$
$$= 0 + 0.2 f(0,0)$$
$$= 0.2[0 + e^0].$$
$$y_1^{(0)} = 0.2.$$

$$y_1^{(1)} = y_0 + \frac{h}{2}\left(f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$$
$$= 0 + \frac{0.2}{2}\left[f(0,0) + f(0.2, 0.2)\right].$$
$$= \frac{0.2}{2}\left[1 + 0.2 + e^{0.2}\right]$$
$$= 0.1\left[1 + 0.2 + 1.2214\right].$$
$$= 0.2421.$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left(f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$$
$$= 0 + \overset{0.1}{\cancel{\frac{0.2}{2}}}\left[1 + 0.2421 + e^{0.2}\right].$$
$$= 0.2463.$$

$$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(2)})\right)$$
$$= 0 + 0.1\left[1 + 0.2463 + e^{0.2}\right]$$
$$= 0.2467.$$

$$y_1^{(4)} = y_0 + \frac{h}{2}\left(f(x_0, y_0) + f(x_1, y_1^{(3)})\right)$$
$$= 0 + 0.2\left(1 + 0.2467 + e^{0.2}\right] = 0.2468.$$

$$y_2^{(5)} = y_0 + \frac{h}{2}\left(f(x_0, y_0) + f(x_1, y_1^{(4)})\right]$$
$$= 0 + 0.1\left(1 + 0.2468 + e^{0.2}\right] = 0.2468.$$

$$\boxed{y_1(0.2) = 0.2468}$$

To find y_2 at $x_2 = x_1 + h = 0.2 + 0.2 = 0.4$

$y_2^{(0)} = y_1 + hf(x_1, y_1)$

$= 0.2468 + 0.2 f(0.2, 0.2468)$

$= 0.2468 + 0.2(0.2468 + e^{0.2})$

$= 0.5404$

$y_2^{(1)} = y_1 + \frac{h}{2} [f(x_1, y_1) + f(x_2, y_2^{(0)})]$

$= 0.2468 + \frac{0.2}{2} [f(0.2, 0.2468) + f(0.4, 0.5404)]$

$= 0.2468 + \frac{0.2}{2} [0.2468 + e^{0.2} + 0.5404 + e^{0.4}]$

$= 0.5968$

$y_2^{(2)} = y_1 + \frac{h}{2} [f(x_1, y_1) + f(x_2, y_2^{(1)})]$

$= 0.2468 + \frac{0.2}{2} [0.2468 + e^{0.2} + 0.5968 + e^{0.4}]$

$= 0.6024$

$y_2^{(3)} = y_1 + \frac{h}{2} [f(x_1, y_1) + f(x_2, y_2^{(2)})]$

$= 0.2468 + \frac{0.2}{2} [0.2468 + e^{0.2} + 0.6024 + e^{0.4}]$

$= 0.6030$

$y_2^{(4)} = y_1 + \frac{h}{2} [f(x_1, y_1) + f(x_2, y_2^{(3)})]$

$= 0.2468 + \frac{0.2}{2} [0.2468 + e^{0.2} + 0.6030 + e^{0.4}]$

$= 0.6031$

$y_2^{(5)} = y_1 + \frac{h}{2} [f(x_1, y_1) + f(x_2, y_2^{(4)})]$

$= 0.2468 + \frac{0.2}{2} [0.2468 + e^{0.2} + 0.6031 + e^{0.4}]$

$\boxed{y_2^{(5)} = 0.6031}$

$\boxed{y_2(0.4) = 0.6031}$

② Solve $\frac{dy}{dx} = x^2+y$, $y(0)=1$ by modified euler's method. and compute $y(0.02)$ & $y(0.04)$

Given $f(x,y) = x^2+y$ $\quad x_0 = 0$ $\quad y_0 = 1$. $h = 0.02$.

By modified euler's method.

$y_1^{(0)} = y_0 + hf(x_0, y_0)$

$= 1 + (0.02)[1]$.

$= 1.02$.

$\left[\begin{aligned} x_1 &= x_0+h. \\ &= 0+0.02. \\ &= 0.02. \end{aligned}\right]$

$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0,y_0) + f(x_1, y_1^{(0)})\right]$

$= 1 + \frac{0.02}{2}\left(1 + (0.02)^2 + 1.02\right)$

$= 1 + 0.01\left(1 + 0.0004 + 1.02\right)$

$= 1.0202$.

$y_1^{(2)} = y_0 + \frac{h}{2}\left(f(x_0, y_0) + f(x_1, y_1^{(1)})\right)$

$= 1 + 0.01\left(1 + (0.02)^2 + 1.0202\right)$

$= 1.0202$.

$y_1(0.02) = 1.0202$

$\boxed{y(0.04)}$ $\quad x_2 = 0.04$. $y_1 = 1.0202$.

$y_2^{(0)} = y_1 + h.f(x_1, y_1)$

$= 1.0202 + 0.02(1.0218)$

$= 1.0406$.

$y_2^{(1)} = y_1 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$

$= 1.0202 + 0.01[1.0218 +$

Runge - kutta method (R-K method)

To solve the ordinary D.E $\frac{dy}{dx} = f(x,y)$, $y(x_0) = y_0$.

by R-K method.

$$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

where - $k_1 = hf(x_0, y_0)$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

Problems:- Using R-K method compute y at $x = 0.1, 0.2$ from $y' + y = 0$. $y(0) = 1$.

$y' = -y$.

Sol $f(x,y) = -y$, $y_0 = 1$, $x_0 = 0$, $h = 0.1$.

By R-K method

$$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

where $k_1 = hf(x_0, y_0)$

$= 0.1\ f(0,1)$

$= -0.1.$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$= 0.1\ f\left(0 + \frac{0.1}{2}, 1 + \frac{(-0.1)}{2}\right)$$

$= -0.095.$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$= 0.1\ f\left(0 + \frac{0.1}{2}, 1 + \frac{(-0.095)}{2}\right)$$

$= -0.0952.$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

$= 0.1\ f(0 + 0.1, 1 + (-0.0952))$

$= -0.0904.$

$y_1(x_1 = x_0 + h = 0.1) = 1 + \frac{1}{6}[-0.1 + 2(-0.095) + 2(-0.0952) + (-0.0904)]$

$y_1 = 0.9048$ at $x_1 = 0.1$.

y_2 at $x_2 = x_1 + h = 0.1 + 0.1 = 0.2$.

$y_2 = y_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$k_1 = hf(x_1, y_1)$
$= 0.1 f(0.1, 0.9048)$
$= -0.0904$.

$k_2 = hf(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2})$
$= 0.1 f(0.1 + \frac{0.1}{2}, 0.9048 + (\frac{-0.0904}{2}))$
$= -0.0859$.

$k_3 = hf(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2})$
$= 0.1 f(0.1 + \frac{0.1}{2}, 0.9048 + (-\frac{0.0859}{2}))$
$= -0.0861$.

$k_4 = hf(x_1 + h, y_1 + k_3)$
$= 0.1 f(0.1 + 0.1, 0.9048 - 0.0861)$

$k_4 = -0.0818$.

$y_2 = y_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$y_2 =$ at $x_2 = 0.2$.

Apply fourth order R-K method, to find an approx value of y when $x = 1.2$, in steps of 0.1, given that $y' = x^2 + y^2$.

$y(1) = 1.5$. $f(x, y) = x^2 + y^2$) $x_0 = 1$ $y_0 = 1.5$.

$h = 0.1$.

By R-K method:

$$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

where $k_1 = h f(x_0, y_0) = 0.1 \times 3.25$

$$k_1 = 0.325$$

$$k_2 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$= 0.1 f(1.05, 1.6625)$$

$$k_2 = 0.3866$$

$$k_3 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$= 0.1 f(1.05, 1.6933)$$

$$k_3 = \cancel{0.6930}\ 0.3969$$

$$k_4 = h f(x_0 + h, y_0 + k_3)$$

$$= 0.1 f(1.1, 1.8969)$$

$$k_4 = 0.4808$$

$(x_1 = x_0 + h = 1.1)$ $y_1 = 1.5 + \frac{1}{6}(0.325 + 2(0.3866) + 2(0.3969) + 0.4808)$

$$\boxed{y_1 = 1.8954}$$

$$y_2 = y_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$k_1 = h f(x_1, y_1)$$

$$= 0.1 f(1.1, 1.8954)$$

$$= 0.4802$$

$k_2 = h f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = 0.1 f(1.15, 2.1355)$

$$= 0.5882$$

$$K_3 = h f\left(x_1 + \frac{h}{2}, y_1 + \frac{K_2}{2}\right)$$

$$= 0.1 f(1.15, 2.1895)$$

$$K_3 = 0.6116.$$

$$K_4 = h f(x_1 + h, y_1 + K_3)$$

$$= 0.1 (1.2, 2.507)$$

$$= 0.7725.$$

$$y_2 = y_1 + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$

$$\boxed{y_2 = 2.5041}$$

③ Using R-K method, find $y(0.2)$ for the equation

$\frac{dy}{dx} = \frac{y-x}{y+x}$, $y(0) = 1$. Take $h = 0.2$. Ans 1.1678.

$f(x,y) = \frac{y-x}{y+x}$. $x_0 = 0$, $y_0 = 1$, $h = 0.2$

$$y_1 = y_0 + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$

$$K_1 = h.f(x_0, y_0) = 0.2 f(0, 1) = 0.2\left(\frac{1-0}{1}\right)$$

$$K_1 = 0.2.$$

$$K_2 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right) = 0.2 f(0.1, 1.1)$$

$$= \left(\frac{1.1 - 0.1}{1.1 + 0.1}\right) 0.2 = \left(\frac{1}{1.2}\right) 0.2.$$

$$K_2 = 0.1666.$$

$$K_3 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right) = 0.2 f(0.1, 1.0833)$$

$$= \left(\frac{1.0833 - 0.1}{1.0833 + 0.1}\right) 0.2 = 0.1661.$$

$$K_4 = h.f(x_0 + h, y_0 + K_3) = 0.2 f(0.2, 1.1661)$$

$$= 0.1414.$$

$$y_1 = y_0 + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4) = 1.1678.$$

(4) Apply fourth order R-K method to find $y(0.1)$ and $y(0.2)$ given $y' = xy + y^2$, $y(0) = 1$

$f(x,y) = xy + y^2 \quad x_0 = 0 \quad y_0 = 1 \quad h = 0.1$

$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$k_1 = hf(x_0, y_0)$

$= 0.1\, f(0,1) = 0.1(1)$

$= 0.1$

$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.1 f(0.05, 1.05)$

$= 0.1155$

$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.1 f(0.05, 1.05775)$

$= 0.1171$

$k_4 = hf(x_0 + h, y_0 + k_3) = 0.1 f(0.1, 1.1171)$

$= 0.1359$

$y_1 = 1 + \frac{1}{6}(0.1 + (0.1155)2 + (0.1171)2 + 0.1359)$

$= 0.1168 + 1 = 1.1168$

$x_1 = x_0 + h = 0 + 0.1 = 0.1 \quad y_1 = 1.1168$

$k_1 = hf(x_1, y_1) = 0.1 f(0.1, 1.1168)$

$k_1 = 0.1358$

$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = 0.1 f(0.15, 1.1847)$

$k_2 = 0.1581$

$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) = 0.1 f(0.15, 1.1958)$

$k_3 = 0.1609$

$k_4 = hf(x_1 + h, y_1 + k_3) = 0.1 f(0.2, 1.2777)$

$k_4 = 0.1888$

$y_2 = 1.1168 + \frac{1}{6}(0.1358 + 2 \times 0.1581 + 2 \times 0.1609 + 0.1888) = 1.2772$

(5) Apply RK method, to find $y(0.2)$ & $y(0.4)$ given $dy/dx = x^2+y^2$ $y(0)=1$ take $h=0.1$

$f(x,y) = x^2+y^2$ $h=0.1$ $x_0=0$, $y_0=1$

By R-K method

$y_1 = y_0 + \frac{1}{6}(k_1+2k_2+2k_3+k_4)$

$k_1 = hf(x_0, y_0) = 0.1 f(0,1)$

$=0.1$

$k_2 = hf(x_0+\frac{h}{2}, y_0+\frac{k_1}{2}) = 0.1 f(0.05, 1.05)$

$=0.1105$

$k_3 = hf(x_0+\frac{h}{2}, y_0+\frac{k_2}{2}) = 0.1 f(0.05, 1.0552)$

$=0.1115$

$k_4 = hf(x_0+h, y_0+k_3) = 0.1 f(0.1, 1.1115)$

$=0.1245$

$y_1 = y_0+\frac{1}{6}(k_1+2k_2+2k_3+k_4)$

$=1.1114$

$x_1 = x_0+h = 0+0.1 = 0.1$ $y_1 = 1.1114$ $h=0.1$

$k_1 = hf(x_1, y_1) = 0.1 f(0.1, 1.1114)$

$=0.1245$

$k_2 = hf(x_1+\frac{h}{2}, y_1+\frac{k_1}{2}) = 0.1 f(0.15, 1.1736)$

$=0.1399$

$k_3 = hf(x_1+\frac{h}{2}, y_1+\frac{k_2}{2}) = 0.1 f(0.15, 1.1813)$

$=0.1417$

$k_4 = hf(x_1+h, y_1+k_3) = 0.1 f(0.2, 1.2531)$

$=0.1610$

$y_2 = y_1+\frac{1}{6}(k_1+2k_2+2k_3+k_4)$

$=1.2528$

(6) Using R-K method find,

(i) $y' = x - 2y$, $y(0) = 1$ taking $h = 0.1$ find y at $x = 0.1, 0.2$

$x_0 = 0 \quad y_0 = 1 \quad h = 0.1$.

$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$k_1 = hf(x_0, y_0) = 0.1 f(0, 1)$

$= -0.2$.

$k_2 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}) = 0.1 f(0.05, 0.9)$

$= -0.175$.

$k_3 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}) = 0.1 f(0.05, 0.9125)$

$= -0.1775$.

$k_4 = hf(x_0 + h, y_0 + k_3) = 0.1 f(0.1, 0.8225)$

$= -0.1545$

$y_1 = 1 + \frac{1}{6}(-0.2 - 2 \times 0.175 - 2 \times 0.1775 - 0.1545)$

$y_1 = 0.8234$.

$x_1 = 0.1 \quad y_1 = 0.8234$.

$y_2 = y_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$k_1 = h \cdot f(x_1, y_1) = 0.1 f(0.1, 0.8234)$

$= -0.1546$.

$k_2 = h \cdot f(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}) = 0.1 f(0.15, 0.7461)$

$= -0.1342$.

$k_3 = h \cdot f(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}) = 0.1 f(0.15, 0.7563)$

$= -0.1362$.

$k_4 = h \cdot f(x_1 + h, y_1 + k_3) = 0.1 f(0.2, 0.6872)$

$= -0.1174$.

$y_2 = 0.8234 - \frac{1}{6}(0.1546 + 2 \times 0.1342 + 2 \times 0.1362 + 0.1174)$

$= 0.6879$//

(ii) $y' = x^2 - y$ $\quad y(0) = 1$ $\quad y(0.1)$ & $(y(0.2))$

$y_1 = y_0 + 1/6(k_1 + 2k_2 + 2k_3 + k_4)$

$k_1 = hf(x_0, y_0) = 0.1 f(0,1)$

$= -0.1$

$k_2 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}) = 0.1 f(0.05, 0.95)$

$= -0.09475$

$k_3 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}) = 0.1 f(0.05, 0.9526)$

$= -0.095$

$k_4 = hf(x_0 + h, y_0 + k_3) = 0.1 f(0.1, 0.905)$

$= -0.0895$

$y_1 = 1 + \frac{1}{6}(-0.1 - 0.1895 - 0.19 - 0.0895)$

$\boxed{y_1 = 0.9052}$

$x_1 = x_0 + h, \quad h = 0.1, \quad y_1 = 0.9052$

$= 0.1$

$k_1 = hf(x_1, y_1) = 0.1 f(0.1, 0.9052)$

$= -0.08952$

$k_2 = hf(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}) = 0.1 f(0.15, 0.8604)$

$= -0.0837$

$k_3 = hf(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}) = 0.1 f(0.15, 0.8633)$

$= -0.0841$

$k_4 = hf(x_1 + h, y_1 + k_3) = 0.1 f(0.2, 0.8211)$

$= -0.0781$

$y_2 = 0.9052 + \frac{1}{6}(-0.0895 - 0.1675 - 0.1682 - 0.0781)$

$\boxed{y_2 = 0.8213}$

(iii) $y' = f(x, y) = 3x + y^2 \quad y(1) = 1.2$ find $y(1.1)$

$f(x, y) = 3x + y^2 \quad x_0 = 1 \quad y_0 = 1.2 \qquad y(1.2) = ?$

$h = 0.1$

$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$k_1 = h \cdot f(x_0, y_0) = 0.1 f(1, 1.2)$

$= 0.444$

$k_2 = h \cdot f(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}) = 0.1 f(1.05, 1.422)$

$= 0.5172$

$k_3 = h \cdot f(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}) = 0.1 f(1.05, 1.4586)$

$= 0.5277$

$k_4 = h \cdot f(x_0 + h, y_0 + k_3) = 0.1 f(1.1, 1.7277)$

$= 0.6284$

$y_1 = 1.2 + \frac{1}{6}(0.444 + 2 \times 0.5172 + 2 \times 0.5277 + 0.6284)$

$\boxed{y_1 = 1.7270}$

(iv) $y' = x - y, \quad y(1) = 0.4 \quad y(1.2) = ?$

$x_0 = 1 \quad y_0 = 0.4 \quad h = 0.2$

$k_1 = hf(x_0, y_0) = 0.2 f(1, 0.4)$

$= 0.12$

$k_2 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}) = 0.2 f(1.1, 0.46)$

$= 0.128$

$k_3 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}) = 0.2 f(1.1, 0.464)$

$= 0.1272$

$k_4 = hf(x_0 + h, y_0 + k_3) = 0.2 f(1.2, 0.5272)$

$= 0.1345$

$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$= 0.4 + \frac{1}{6}(0.12 + 2 \times 0.128 + 2 \times 0.1272 + 0.134$

$y(1.2) = y_1 = 0.5273$ //

Module-3

FOURIER TRANSFORMS

Definition :-

① Let $F(x)$ is a function defined on $[-\infty,\infty]$, then the Fourier transform can be defined as

$$F[F(x)] = \int_{-\infty}^{\infty} e^{isx} F(x)\,dx = f(s)$$

where 'F' is called the Fourier transform operator and 's' be the parameter either real (or) complex.

The inverse Fourier transform is defined as

$$F^{-1}[f(s)] = F(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isx} f(s)\,ds$$

② Let $F(x)$ be a function defined on $[0,\infty]$, then the Fourier cosine transform of $F(x)$ is defined as

$$F_c[F(x)] = \int_{0}^{\infty} F(x)\cos(sx)\,dx = f_c(s).$$

and its inverse fourier cosine transform is

$$F^{-1}[f_c(s)] = F(x) = \frac{2}{\pi}\int_{0}^{\infty} f_c(s)\cos(sx)\,ds$$

Similarly, the fourier sin transform of $F(x)$ is $F_s[F(x)] = \int_{0}^{\infty} F(x)\sin(sx)\,dx = f_s(s)$ and its inverse sin transform is $F^{-1}[f_s(s)] = F(x) = \frac{2}{\pi}\int_{0}^{\infty} f_s(s)\sin(sx)\,ds.$

① Find the fourier transform of $e^{-a^2x^2}$, $a>0$ and hence Show that the fourier transform of $e^{-x^2/2}$ is $\sqrt{2\pi}\,e^{-s^2/2}$

Given $F(x) = e^{-a^2x^2}$, $a > 0$

WKT

$$F[F(x)] = \int_{-\infty}^{\infty} e^{isx} F(x)\, dx$$

$$= \int_{-\infty}^{\infty} e^{-a^2x^2} e^{isx}\, dx$$

$$= \int_{-\infty}^{\infty} e^{-a^2x^2 + isx}\, dx$$

$$= \int_{-\infty}^{\infty} e^{-(a^2x^2 - isx)}\, dx$$

$$= \int_{-\infty}^{\infty} e^{-\left[(ax)^2 - 2(ax)\left(\frac{is}{2a}\right) + \left(\frac{is}{2a}\right)^2 - \left(\frac{is}{2a}\right)^2\right]} dx$$

$$= \int_{-\infty}^{\infty} e^{-\left\{\left(ax - \frac{is}{2a}\right)^2 - \frac{i^2s^2}{4a^2}\right\}} dx$$

$$= \int_{-\infty}^{\infty} e^{-\left\{\left(ax - \frac{is}{2a}\right)^2 + \frac{s^2}{4a^2}\right.} dx$$

$$= \int_{-\infty}^{\infty} e^{-\left(ax - \frac{is}{2a}\right)^2} e^{-s^2/4a^2}\, dx$$

$$= e^{-s^2/4a^2} \int_{-\infty}^{\infty} e^{-\left(ax - \frac{is}{2a}\right)^2} dx$$

Let $ax - \frac{is}{2a} = u$

d.w.r.t 'x'

$a\, dx = du$

$dx = \frac{1}{a} du$

when $x = \infty \Rightarrow u = \infty$

$x = -\infty \Rightarrow u = -\infty$

$$F[F(x)] = e^{-s^2/4a^2} \int_{-\infty}^{\infty} e^{-u^2} \cdot \frac{1}{a}\, du$$

$$= \frac{1}{a} e^{-s^2/4a^2} \int_{-\infty}^{\infty} e^{-u^2}\, du$$

$$= \frac{1}{a} e^{-s^2/4a^2} \sqrt{\pi}$$

$$F[e^{-a^2x^2}] = \frac{\sqrt{\pi}}{a} e^{-s^2/4a^2} \quad \text{---} (1)$$

Let $a = \frac{1}{\sqrt{2}}$

$$(1) \Rightarrow F[e^{-x^2/2}] = \frac{\sqrt{\pi}}{\frac{1}{\sqrt{2}}} \cdot e^{-s^2/4\left(\frac{1}{2}\right)}$$

$$F[e^{-x^2/2}] = \sqrt{2\pi}\, e^{-s^2/2}$$

(2) Find the Fourier transform of $F(x) = \begin{cases} 1, & |x| < a \\ 0, & |x| \geq a \end{cases}$ and hence find $\int_0^\infty \frac{\sin x}{x}\, dx$

Given $F(x) = \begin{cases} 1, & |x| < a \\ 0, & |x| \geq a \end{cases}$

$$F(x) = \begin{cases} 1, & -a < x < a \\ 0, & x \geq a \end{cases}$$

WKT $F[F(x)] = \int_{-\infty}^{\infty} F(x) e^{isx}\, dx$

$$= \int_{-\infty}^{-a} F(x) e^{isx}\, dx + \int_{-a}^{a} F(x) e^{isx}\, dx + \int_{a}^{\infty} F(x) e^{isx}\, dx$$

$$= 0 + \int_{-a}^{a} 1\, e^{isx}\, dx + 0$$

$$F[F(x)] = \int_{-a}^{a} e^{isx}\, dx$$

$$= \left[\frac{e^{isx}}{is}\right]_{-a}^{a}$$

$$= \frac{1}{is}\left[e^{ias} - e^{-ias}\right]$$

$$= \frac{1}{is}\left[\cos(as) + i\sin(as) - \cos(as) + i\sin(as)\right]$$

$$= \frac{1}{\cancel{i}s}\left[2\cancel{i}\sin(as)\right]$$

$$f(s) = \frac{2}{s} \sin as$$

WKT

$$F^{-1}[f(s)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(s)\, e^{-isx}\, dx = F(x)$$

$$= \int_{-\infty}^{\infty} \frac{2}{s}\sin(as)\, e^{-isx} dx = 2\pi\, F(x) = 2\pi \begin{cases} 1, & a < x < a \\ 0 & x \geq a \end{cases}$$

Let $x = 0$

$$① \Rightarrow \int_{-\infty}^{\infty} \frac{2}{s}\sin(as)\, ds = 2\pi(1)$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{2}{s}\sin(as)\, ds = 2\pi(1)$$

$$\Rightarrow 2\int_{-\infty}^{\infty} \frac{\sin as}{s}\, ds = \pi$$

$$= \int_{0}^{\infty} \frac{\sin as}{s}\, ds = \frac{\pi}{2} \quad -②$$

when $a = 1$

$$\int_{0}^{\infty} \frac{\sin s}{s}\, ds = \frac{\pi}{2}$$

take $s = x$

$$\int_{0}^{\infty} \frac{\sin x}{x}\, dx = \frac{\pi}{2}$$

③ Find the fourier transform of $F(x) = \begin{cases} 1, & |x| < 1 \\ 0, & |x| \geq 1 \end{cases}$ and hence find $\int_{0}^{\infty} \frac{\sin x}{x}\, dx$

Given

$$F(x) = \begin{cases} 1, & |x| < 1 \\ 0, & |x| \geq 1 \end{cases}$$

$$F(x) = \begin{cases} 1, & -1 < x < 1 \\ 0, & x \geq 1 \end{cases}$$

WKT $F[F(x)] = \int_{-\infty}^{\infty} F(x)\, e^{isx}\, dx$

$$= \int_{-\infty}^{-1} F(x)\, e^{isx} dx + \int_{-1}^{1} F(x)\, e^{isx}\, dx + \int_{1}^{\infty} F(x)\, e^{isx}\, dx$$

$$= 0 + \int_{-1}^{1} F(x) e^{isx} dx + 0$$

$$F[f(x)] = \int_{-1}^{1} e^{isx} dx$$

$$= \left[\frac{e^{isx}}{is}\right]_{-1}^{1}$$

$$= \frac{1}{is}\left[e^{is} - e^{-is}\right]$$

$$= \frac{1}{is}\left[\cos(s) + i\sin(s) - \cos(s) + i\sin(s)\right]$$

$$= \frac{1}{\cancel{i}s}\left[2\cancel{i}\sin(s)\right]$$

$$= \frac{2}{s}[\sin s]$$

WKT

$$F^{-1}[f(s)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(s) e^{-isx} ds = F(x)$$

$$= \int_{-\infty}^{\infty} \frac{2}{s} \sin s \, e^{-isx} ds = 2\pi F(x) = 2\pi \begin{cases} 1, & -1 < x < 1 \\ 0, & x > 1 \end{cases}$$

Let $x = 0$

$$(7) \Rightarrow \int_{-\infty}^{\infty} \frac{2}{s} \sin ds = 2\pi(1)$$

$$= \int_{-\infty}^{\infty} \frac{\cancel{2}}{s} \sin s \, ds = \cancel{2}\pi(1)$$

$$= 2\int_{-\infty}^{\infty} \frac{\sin s}{s} ds = \pi$$

$$= \int_{0}^{\infty} \frac{\sin s}{s} ds = \frac{\pi}{2}$$

when $s = x$

$$= \int_{0}^{\infty} \frac{\sin x}{x} dx = \pi/2 \quad //$$

(4) Find the fourier transform of $e^{-|x|}$

$$F(x) = \begin{cases} e^{-(-x)}, & x < 0 \\ e^{-x}, & x > 0 \end{cases}$$

$$F(x) = \begin{cases} e^{x}, & x < 0 \\ e^{-x} & x > 0 \end{cases}$$

WKT

$$F[F(x)] = \int_{-\infty}^{\infty} F(x) e^{isx} dx.$$

$$= \int_{-\infty}^{0} F(x) e^{isx} dx + \int_{0}^{\infty} F(x) e^{isx} dx$$

$$= \int_{-\infty}^{0} e^{x} e^{isx} dx + \int_{0}^{\infty} e^{-x} e^{isx} dx$$

$$= \int_{-\infty}^{0} e^{x+isx} dx + \int_{0}^{\infty} e^{-x+isx} dx$$

$$= \int_{-\infty}^{0} e^{[1+is]x} dx + \int_{0}^{\infty} e^{-[1-is]x} dx$$

$$= \left[\frac{e^{(1+is)x}}{1+is}\right]_{-\infty}^{0} + \left[\frac{e^{-(1-is)x}}{1-is}\right]_{0}^{\infty}$$

$$= \frac{1}{1+is}\left[e^{(1+is)x}\right]_{-\infty}^{0} - \frac{1}{1-is}\left[e^{-(1-is)x}\right]_{0}^{\infty}$$

$$= \frac{1}{1+is}[1-0] - \frac{1}{1-is}[0-1]$$

$$= \frac{1}{1+is} + \frac{1}{1-is}$$

$$= \frac{1-\cancel{is}+1+\cancel{is}}{1^2+i^2s^2}$$

$$= \frac{2}{1+s^2} \quad //$$

(5) Find the fourier transform $F(x) = \begin{cases} a^2 - x^2, & |x| \le a \\ 0, & |x| > a \end{cases}$ hence show that (i) $\int_{0}^{\infty} \frac{\sin x - x\cos x}{x^3} dx = \frac{\pi}{4}$

(ii) $\int_0^\infty \frac{x\cos x - \sin x}{x^3} \cos\left(\frac{x}{2}\right) dx = \frac{-3\pi}{16}$

Given $F(x) = \begin{cases} a^2 - x^2, & -a \le x \le a \\ 0, & x > a \end{cases}$

WKT $F[F(x)] = \int_{-\infty}^{\infty} e^{isx} F(x)\,dx$

$= \int_{-\infty}^{a} e^{isx} F(x)\,dx + \int_{-a}^{a} e^{isx} F(x)\,dx + \int_{a}^{\infty} e^{isx} F(x)\,dx$

$= \int_{-a}^{a} (a^2 - x^2) e^{isx}\,dx$

$= (a^2 - x^2)\int_{-a}^{a} e^{isx}\,dx - \int_{-a}^{a}\left[-2x \int e^{isx}\,dx\right] dx$

$= \frac{1}{is}(0-0) + \frac{2}{is}\int_{-a}^{a} x\, e^{isx}\,dx$

$= \frac{2}{is}\left\{ x\int_{-a}^{a} e^{isx}\,dx - \int_{-a}^{a}\left[1 \cdot \int e^{isx}\,dx\right] dx \right\}$

$= \frac{2}{is}\left\{ \frac{1}{is}\left[x e^{isx}\right]_{-a}^{a} - \frac{1}{i^2 s^2}\left[e^{isx}\right]_{-a}^{a} \right\}$

$= \frac{2}{i^2 s^2}\left[x e^{isx}\right]_{-a}^{a} - \frac{2}{i^3 s^3}\left[e^{isx}\right]_{-a}^{a}$

$= -\frac{2}{s^2}\left[a e^{ias} + a e^{-ias}\right] + \frac{2}{is^3}\left[e^{ias} - e^{-ias}\right]$

$= -\frac{2a}{s^2}\left[e^{ias} + e^{-ias}\right] + \frac{2}{is^3}\left[e^{ias} - e^{-ias}\right]$

$= -\frac{2a}{s^2}\left[2\cos(as)\right] + \frac{2}{is^3}\left[2i\sin(as)\right]$

$= \frac{-4a\cos(as)}{s^2} + \frac{4\sin(as)}{s^3}$

$= \frac{4\sin(as) - 4as\cos(as)}{s^3}$

$f(s) = \frac{4}{s^3}\left[\sin(as) - as\cos(as)\right]$

We know that

The fourier inverse transform is

$$F^{-1}[f(s)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isx} f(s)\,ds = F(x)$$

$$\Rightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isx}\frac{4}{s^3}[\sin(as) - as\cos(as)ds] = \begin{cases} a^2 - x^2, & |x| \le a \\ 0, & |x| > a \end{cases}$$

$$\Rightarrow \int_{-\infty}^{\infty} e^{-isx}\frac{[\sin(as) - as\cos(as)]}{s^3}\,ds = \frac{\pi}{2}\begin{cases} a^2 - x^2, & |x| \le a \\ 0, & |x| > a \end{cases}$$

$$\Rightarrow \int_{-\infty}^{\infty} [\cos(sx) - i\sin(sx)]\frac{[\sin(as) - as\cos(as)]}{s^3}\,ds =$$

$$\frac{\pi}{2}\begin{cases} a^2 - x^2, & |x| \le a \\ 0, & |x| > a \end{cases}$$

But $x = 0$

$$① \Rightarrow \int_{-\infty}^{\infty} \frac{\sin(as) - as\cos(as)}{s^3}\,ds = \frac{\pi}{2}a^2 \quad -②$$

$$\int_{-\infty}^{\infty} \frac{\sin s - s\cos s}{s^3}\,ds = \frac{\pi}{2}, \quad \text{for } a = 1$$

$$\Rightarrow 2\int_{0}^{\infty} \frac{\sin s - s\cos s}{s^3}\,ds = \frac{\pi}{2}$$

$$\int_{0}^{\infty} \frac{\sin s - s\cos s}{s^3}\,ds = \frac{\pi}{4}$$

$$\int_{0}^{\infty} \frac{\sin x - x\cos x}{x^3}\,dx = \frac{\pi}{4} \quad \text{for } s = x.$$

(ii) Let $x = a/2$

$$① \Rightarrow \int_{-\infty}^{\infty} \frac{\sin(as) - s\cos(as)}{s^3}\cos\left(\frac{as}{2}\right) = \frac{\pi}{2}\left(a^2 - \frac{a^2}{4}\right) = \frac{3\pi a^2}{8}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\sin s - s\cos s}{s^3} \cos\left(\frac{s}{2}\right) ds = \frac{3\pi}{8} \text{ for } a = 1$$

$$\Rightarrow 2\int_{0}^{\infty} \frac{\sin s - s\cos s}{s^3} \cos\left(\frac{s}{2}\right) ds = \frac{3\pi}{8}$$

$$\Rightarrow \int_{0}^{\infty} \frac{\sin s - s\cos s}{s^3} \cos\left(\frac{s}{2}\right) ds = \frac{3\pi}{16}$$

$$\Rightarrow \int_{0}^{\infty} \frac{s\cos s - \sin s}{s^3} \cos\left(\frac{s}{2}\right) ds = \frac{-3\pi}{16}$$

$$\Rightarrow \int_{0}^{\infty} \frac{x\sin x - \sin x}{x^3} \cos\left(\frac{x}{2}\right) dx = \frac{-3\pi}{16} \;//$$

⑥ Find the fourier transform $F(x) = \begin{cases} 1-x^2, & |x| \le 1 \\ 0, & |x| > 1 \end{cases}$

hence show that (i) $\int_{0}^{\infty} \frac{\sin x - x\cos x}{x^3} dx = \frac{\pi}{4}$

(ii) $\int_{0}^{\infty} \frac{x\cos x - \sin x}{x^3} \cos\left(\frac{x}{2}\right) dx = \frac{-3\pi}{16}$

Given $F(x) \begin{cases} 1-x^2, & -1 \le x \le 1 \\ 0, & x > 1 \end{cases}$

WKT

$$F[F(x)] = \int_{-\infty}^{\infty} e^{isx} F(x)\, dx$$

$$= \int_{-\infty}^{-1} e^{isx} F(x)\, dx + \int_{-1}^{1} e^{isx} F(x)\, dx + \int_{1}^{\infty} e^{isx} F(x)\, dx$$

$$= \int_{-1}^{1} (1^2 - x^2) e^{isx}\, dx$$

$$= (1-x^2) \int_{-1}^{1} e^{isx} dx - \int_{-1}^{1} \left[-2x \int e^{isx} dx\right] dx$$

$$= \frac{1}{is}\left[(1-x^2) e^{isx}\right]_{-1}^{1} + \frac{2}{is}\int_{-1}^{1} x e^{isx}\, dx$$

$$= \frac{1}{is}(0-0) + \frac{2}{is}\int_{-1}^{1} x e^{isx}\, dx$$

$$= \frac{2}{is}\left\{ x \int_{-1}^{1} e^{isx}\, dx - \int_{-1}^{1} \left[1 \int e^{isx} dx\right] dx \right\}$$

$$= \frac{2}{is}\left\{\frac{1}{is}\left[xe^{isx}\right]_{-1}^{1} - \frac{1}{i^2s^2}\left[e^{isx}\right]_{-1}^{1}\right\}$$

$$= \frac{2}{i^2s^2}\left[xe^{isx}\right]_{-1}^{1} - \frac{2}{i^2s^2}\left[e^{isx}\right]_{-1}^{1}$$

$$= -\frac{2}{s^2}\left[e^{is}+e^{-is}\right] + \frac{2}{is^3}\left[e^{is}-e^{-is}\right]$$

$$= \frac{-2}{s^2}\left[2\cos(s)\right] + \frac{2}{is^3}\left[2i\sin(s)\right]$$

$$= \frac{-4\cos(s)}{s^2} + \frac{4\sin(s)}{s^3}$$

$$= \frac{4\sin(s) - 4s\cos(s)}{s^3}$$

$$f(s) = \frac{4}{s^3}\left[\sin(s) - s\cos(s)\right]$$

WKT

The Fourier inverse transform is

$$F^{-1}[f(s)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isx} f(s)\,ds = F(x)$$

$$\Rightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isx}\frac{4}{s^3}\left[\sin s - s\cos s\right]ds = \begin{cases} 1-x^2, & |x|\le 1 \\ 0, & |x|>1 \end{cases}$$

$$\Rightarrow \int_{-\infty}^{\infty} e^{-isx}\left[\frac{\sin s - s\cos s}{s^3}\right]ds = \frac{\pi}{2}\begin{cases} 1-x^2, & |x|\le 1 \\ 0, & |x|>1 \end{cases}$$

$$\Rightarrow \int_{-\infty}^{\infty}\left[\cos(sx) - i\sin(sx)\right]\frac{\left[\sin(s) - s(\cos s)\right]}{s^3}ds = \frac{\pi}{2}\begin{cases} 1-x^2, & |x|\le 1 \\ 0, & |x|>1 \end{cases}$$

$$\Rightarrow \int_{-\infty}^{\infty}\frac{\sin s - s\cos s}{s^3}\cos(sx)\,ds = \frac{\pi}{2}\begin{cases} 1-x^2, & |x|\le 1 \\ 0, & |x|>1 \end{cases} \quad -(1)$$

But $x = 0$

$$\therefore (1) \Rightarrow \int_{-\infty}^{\infty}\frac{\sin s - s\cos s}{s^3}ds = \frac{\pi}{2} \quad -(2)$$

$$\Rightarrow \int_{-\infty}^{\infty}\frac{\sin s - s\cos s}{s^3}ds = \frac{\pi}{2}$$

$$\Rightarrow 2\int_{0}^{\infty}\frac{\sin s - s\cos s}{s^3}ds = \frac{\pi}{2}$$

$$\Rightarrow \int_0^\infty \frac{\sin s - s\cos s}{s^3}\, ds = \frac{\pi}{4}$$

$$\Rightarrow \int_0^\infty \frac{\sin x - x\cos x}{x^3}\, ds = \frac{\pi}{4} \quad \text{for } s = x$$

(ii) Let $x = \frac{1}{2}$

$$① \Rightarrow \int_{-\infty}^{\infty} \frac{\sin s - s\cos s}{s^3} \cos\left(\frac{s}{2}\right) ds = \frac{\pi}{2}\left(1^2 - \frac{1^2}{4}\right) = \frac{3\pi}{8}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\sin s - s\cos s}{s^3} \cos\left(\frac{s}{2}\right) ds = \frac{3\pi}{8}$$

$$\Rightarrow 2\int_0^\infty \frac{\sin s - s\cos s}{s^3} \cos\left(\frac{s}{2}\right) ds = \frac{3\pi}{8}$$

$$\Rightarrow \int_0^\infty \frac{\sin s - s\cos s}{s^3} \cos\left(\frac{s}{2}\right) ds = \frac{3\pi}{16}$$

$$\Rightarrow \int_0^\infty \frac{s\cos s - \sin s}{s^3} \cos\left(\frac{s}{2}\right) ds = -\frac{3\pi}{16}$$

$$\Rightarrow \int_0^\infty \frac{x\cos x - \sin x}{x^3} \cos\left(\frac{x}{2}\right) dx = \frac{-3\pi}{16} \quad \text{for } s = x$$

7 Find the fourier transform of $F(x) = \begin{cases} 1 - |x|, & |x| \le 1 \\ 0 & |x| > 1 \end{cases}$

$$F(x) = \begin{cases} 1 - |x|, & -1 \le x \le 1 \\ 0, & x > 1 \end{cases}$$

$$F(x) = \begin{cases} 1 + x, & -1 \le x < 0 \\ 1 - x, & 0 \le x < 1 \\ 0, & x > 1 \end{cases}$$

WKT

$$F[F(x)] = \int_{-\infty}^{\infty} F(x) e^{isx} dx$$

$$f(S) = \int_{-\infty}^{-1} F(x) e^{isx} dx + \int_{-1}^{0} F(x) e^{isx} dx + \int_0^1 F(x) e^{isx} dx + \int_1^\infty F(x) e^{isx} dx$$

$$= \int_{-1}^{0} (1+x) e^{isx} dx + \int_0^1 (1-x) e^{isx} dx \quad —①$$

$$\int_{-1}^{0} (1+x) e^{isx} dx = \left\{ (1+x) \int_{-1}^{0} e^{isx} dx - \int_{-1}^{0} \left[1 \int e^{isx} dx\right] dx \right\}$$

$$= \frac{1}{is}\left[(1+x)e^{isx}\right]_{-1}^{0} - \frac{1}{is^2}\left(e^{isx}\right)_{-1}^{0}$$

$$= \frac{1}{is}\left[(1+x)e^{isx}\right]_{-1}^{0} + \frac{1}{s^2}\left(e^{isx}\right)_{-1}^{0}$$

$$= \frac{1}{is}[1-0] + \frac{1}{s^2}[1-e^{-is}]$$

$$= \frac{1}{is} + \frac{1}{s^2}[1-e^{-is}]$$

$$= \int_0^1 (1-x)e^{isx}\,dx = \frac{1}{is}\left[(1-x)e^{isx}\right]_0^1 - \frac{1}{s^2}\left[e^{isx}\right]_0^1$$

$$= \frac{1}{is}[0-1] - \frac{1}{s^2}[e^{is}-1]$$

$$= -\frac{1}{is} - \frac{1}{s^2}[e^{is}-1]$$

$$f(s) = \cancel{\frac{1}{is}} + \frac{1}{s^2}[1-e^{-is}] - \cancel{\frac{1}{is}} - \frac{1}{s^2}[e^{is}-1]$$

$$f(s) = \frac{1}{s^2}[1-e^{-is}-e^{is}+1]$$

$$= \frac{1}{s^2}[2-(e^{is}+e^{-is})]$$

$$= \frac{1}{s^2}[2-2\cos s]$$

$$= \frac{2}{s^2}[1-\cos s]$$

WKT

$$F^{-1}[F(s)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(s)e^{-isx}\,ds = F(x)$$

$$= \int_{-\infty}^{\infty} \frac{2}{s^2}[1-\cos s]e^{-isx}\,dx = 2\pi \begin{cases} 1-|x|, & -1 \le x \le 1 \\ 0, & x > 1 \end{cases} \quad -①$$

Let $x = 0$

$$① \Rightarrow \cancel{2}\int_{-\infty}^{\infty} \frac{1-\cos s}{s^2}\,ds = 2\pi(1) = \cancel{2}\pi$$

$$= \int_{-\infty}^{\infty} \frac{1-\cos s}{s^2}\,ds = \pi$$

$$= \int_{-\infty}^{\infty} \frac{2\sin^2(s/2)}{s^2}\, ds = \pi$$

$$= 2\int_{-\infty}^{\infty} \frac{\sin^2(s/2)}{4\times(s/2)^2}\, ds = \pi$$

$$= \int_{-\infty}^{\infty} \frac{\sin^2(s/2)}{(s/2)^2}\, ds = 2\pi \quad \text{—(2)}$$

Let $\frac{s}{2} = u \Rightarrow s = 2u \Rightarrow ds = 2du$

$$(2) \Rightarrow \int_{-\infty}^{\infty} \frac{\sin^2 u}{u^2}\, \cancel{2}du = \cancel{2}\pi$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\sin^2 u}{u^2}\, du = \pi$$

$$\Rightarrow 2\int_{0}^{\infty} \frac{\sin^2 u}{u^2}\, du = \pi$$

$$\Rightarrow \int_{0}^{\infty} \frac{\sin^2 u}{u^2}\, du = \frac{\pi}{2}$$

$$\Rightarrow \int_{0}^{\infty} \frac{\sin^2 t}{t^2}\, dt = \frac{\pi}{2} \quad \text{for } u = t$$

(8) Find the inverse fourier transform of e^{-s^2}.

Let $f(s) = e^{-s^2}$

$$F(x) = F^{-1}[f(s)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isx} f(s)\, ds$$

$$F(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isx} e^{-s^2}\, ds$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-(s^2+isx)}\, ds$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-(s^2 + 2(s)\left(\frac{ix}{2}\right) + \left(\frac{ix}{2}\right)^2 - \left(\frac{ix}{2}\right)^2]}\, ds$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\left[(s+\frac{ix}{2})^2 - \frac{i^2x^2}{4}\right]} ds$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\left[(s+\frac{ix}{2})^2 + \frac{x^2}{4}\right]} ds$$

$$F(x) = \frac{e^{-x^2/4}}{2\pi} \int_{-\infty}^{\infty} e^{-(s+\frac{ix}{2})^2} ds$$

Let $s + \frac{ix}{2} = u$

$\Rightarrow ds = du$

$$\therefore (1) \Rightarrow F(x) = \frac{e^{-x^2/4}}{2\pi} \int_{-\infty}^{\infty} e^{-u^2} du$$

$$F(x) = \frac{e^{-x^2/4}}{2\pi} \sqrt{\pi}$$

$$F(x) = \frac{1}{2\sqrt{\pi}\cancel{\sqrt{\pi}}} e^{-x^2/4} \cancel{\sqrt{\pi}}$$

$$\Rightarrow F(x) = \frac{1}{2\sqrt{\pi}} e^{-x^2/4}$$

(9) Find the fourier cosine & sin transform of $F(x) = \begin{cases} x, & 0<x<1 \\ 2-x, & 1<x<2 \\ 0, & x>2 \end{cases}$

Given $F(x) = \begin{cases} x, & 0<x<1 \\ 2-x, & 1<x<2 \\ 0, & x>2 \end{cases}$

WKT the fourier sin transform of $F(x)$ is

$$f_s[F(s)] = \int_0^{\infty} F(x) \sin(sx)\, dx$$

$$f_s(s) = \int_0^1 F(x)\sin(sx)dx + \int_1^2 F(x)\sin(sx)dx + \int_2^{\infty} F(x)\sin(sx)dx$$

$$= \int_0^1 x\sin(sx)dx + \int_1^2 (2-x)\sin(sx)\, dx$$

$$f_s(s) = I_1 + I_2$$

$$I_1 = \int_0^1 x \sin(sx) dx$$

$$= x\int_0^1 \sin(sx)\,dx - \int_0^1 \left[1 \int \sin(sx)\,dx\right]dx$$

$$= -\left[\frac{x\cos sx}{s}\right]_0^1 + \frac{1}{s^2}\left[\sin sx\right]_0^1$$

$$= -\left[\frac{\cos s - 0}{s}\right] + \frac{1}{s^2}\left[\sin s - 0\right]$$

$$I_1 = \frac{-\cos s}{s} + \frac{1}{s^2}\sin s$$

$$I_2 = \int_1^2 (2-x)\sin(sx)\,dx$$

$$= \left\{(2-x)\int_1^2 \sin(sx)\,dx - \int_1^2 \left[-1\int \sin(sx)\,dx\right]dx\right\}$$

$$= \left[\frac{(2-x)\cos sx}{s}\right]_1^2 - \frac{1}{s}\int_1^2 \cos sx\,dx$$

$$= \left[\frac{(2-x)\cos sx}{s}\right]_1 \int_1^2 \frac{1}{s^2}\left[\sin sx\right]_1^2$$

$$= -\left[\frac{0 - \cos s}{s}\right] - \frac{1}{s^2}\left[\sin 2s - \sin s\right]$$

$$I_2 = \frac{\cos s}{s} - \frac{1}{s^2}\left[\sin 2s + \frac{1}{s^2}\left[\sin s\right]\right]$$

$$f_s(s) = -\frac{\cancel{\cos s}}{\cancel{s}} + \frac{1}{s^2}\sin s + \frac{\cancel{\cos s}}{\cancel{s}} - \frac{1}{s^2}\left[\sin 2s + \frac{1}{s^2}\sin s\right]$$

$$f_s(s) = \frac{2}{s^2}\sin s - \frac{1}{s^2}\sin 2s \;//$$

WKT the fourier cosin transform of $F(x)$ is

$$F_c[F(x)] = \int_0^\infty F(x)\cos(sx)\,dx$$

$$f_c(s) = \int_0^1 F(x)\cos(sx)\,dx + \int_1^2 F(x)\cos(sx)\,dx + \int_2^\infty F(x)\cos(sx)\,dx$$

$$= \int_0^1 x\cos(sx)\,dx + \int_1^2 (2-x)\cos(sx)\,dx$$

$$f_c(s) = I_1 + I_2$$

$$I_1 = \int_0^1 x\cos(sx)\,dx$$

$$= x\int_0^1 \cos(sx)\,dx - \int_0^1 \left[1 \int \cos(sx)\,dx\right] dx$$

$$= \left[\frac{x\sin sx}{s}\right]_0^1 + \frac{1}{s^2}\,\cos sx\Big]_0^1$$

$$= \frac{\sin s}{s} - 0 + \frac{1}{s^2}\left[\cos sx - 1\right]$$

$$= \frac{\sin s}{s} + \frac{1}{s^2}\cos sx - \frac{1}{s^2}$$

$$I_2 = \int_1^2 (2-x)\cos(sx)\,dx$$

$$= (2-x)\int_1^2 \cos(sx)\,dx - \int_1^2 \left[(-1)\int \cos(sx)\,dx\right] dx$$

$$= -\left[\frac{(2-x)\sin sx}{s}\right]_1^2 - \int_1^2 \frac{1}{s^2}\,\sin sx\Big]_1 dx$$

$$= \frac{x-2}{s}\left[\sin sx\right]_1^2 - \frac{1}{s^2}\left[\cos sx\right]_1^2$$

$$= \frac{\sin s}{s} - \frac{\cos 2s}{s^2} + \frac{\cos s}{s^2}$$

$$F_c[F(x)] = \frac{1}{s}\sin s + \frac{1}{s^2}\cos s - \frac{1}{s^2} + \frac{\sin s}{s} - \frac{\cos 2s}{s^2} + \frac{\cos s}{s^2}$$

$$= \frac{2}{s}\sin s + \frac{2}{s^2}\cos s - \frac{1}{s^2} + \frac{\cos 2s}{s^2} \quad //$$

(10) Find the Fourier sin and cosin of $F(x) = \begin{cases} 4x, & 0<x<1 \\ 4-x, & 1\le x<4 \\ 0, & x\ge 4 \end{cases}$

Given $F(x) = \begin{cases} 4x, & 0<x<1 \\ 4-x, & 1\le x<4 \\ 0, & x\ge 4 \end{cases}$

WKT the fourier cosin transform of $F(x)$ is

$$F_c[F(x)] = \int_0^\infty F(x)\cos(sx)\,dx$$

$$= \int_0^1 F(x)\cos(sx)\,dx + \int_1^4 F(x)\cos(sx)\,dx + \int_4^\infty F(x)\cos(sx)\,dx$$

$$= 4\int_0^1 x\cos(sx)\,dx + \int_1^4 (4-x)\cos(sx)\,dx \quad -(1)$$

$$= \frac{4x\sin(sx)}{s} + \frac{4}{s^2}\left[\cos(sx)\right]_0^1$$

$$= \frac{4}{s}\left[4x\sin(sx)\right]_0^1 + \frac{4}{s}\left[\cos(sx)\right]_0^1$$

$$= \frac{4}{s}\left[\sin(s) - 0\right] + \frac{4}{s^2}\left[\cos(s) - 1\right]$$

$$= \frac{4}{s}\sin s + \frac{4}{s^2}\cos s - \frac{4}{s^2}$$

$$\int_1^4 (4-x)\cos(sx)\,dx = (4-x)\int_1^4 \cos(sx)\,dx - \int_1^4 -1\left[\int\cos(sx)\,dx\right]dx$$

$$= \frac{1}{s}\left[(4-x)\sin(sx)\right]_1^4 - \frac{1}{s^2}\left(\cos(sx)\right)_1^4$$

$$= \frac{1}{s^2}\cos s - \frac{1}{s^2}\cos 4s - \frac{1}{s}\sin s$$

$$f_c(s) = \frac{1}{s}\sin s + \frac{4}{s^2}\cos s - \frac{4}{s^2} + \frac{1}{s^2}\cos - \frac{1}{s^2}\cos 4s - \frac{3}{s}\sin s$$

$$f_c(s) = \frac{1}{s}\sin s + \frac{5}{s^2}\cos s - \frac{1}{s^2}\cos 4s - \frac{4}{s^2}$$

(ii) $F_s[F(x)] = \int_0^\infty F(x)\sin(sx)\,dx$

$$= \int_0^1 F(x)\sin(sx)\,dx + \int_1^4 F(x)\sin(sx)\,dx + \int_4^\infty F(x)\sin(sx)\,dx$$

$$= \int_0^1 4x\sin(sx)\,dx + \int_1^4 (4-x)\sin(sx)\,dx$$

$$= 4x\int_0^1 \sin(sx)\,dx - \int_0^1 4\left[\int\sin(sx)\,dx\right]dx$$

$$= -\frac{1}{s}\left[4x\cos(sx)\right]_0^1 + \frac{4}{s^2}\left[\sin(sx)\right]_0^1$$

$$= -\frac{4}{s}\left[\cos(s) - 1\right] + \frac{4}{s^2}\left[\sin(s) - 0\right]$$

$$= -\frac{4}{s}\cos(s) + \frac{4}{s^2}\sin(s)$$

$$= (4-x)\int_1^4 \sin(sx)\,dx - \int_1^4 -1\left[\int\sin(sx)\,dx\right]dx$$

$$= \frac{-1}{s}\left[(4-x)\cos(sx)\right]_1^4 + \frac{1}{s^2}\left[\sin sx \, dx\right]_1^4$$

$$= \frac{-1}{s}\left[0-3\cos(x)\right] - \frac{1}{s^2}\left[\sin(4s)-\sin(s)\right]$$

$$= \frac{3\cos s}{s} - \frac{1}{s^2}\sin(4s) + \frac{1}{s^2}\sin(s)$$

$$= -\frac{4}{s}\cos(s) + \frac{4}{s^2}\sin(s) + \frac{3\cos s}{s} - \frac{1}{s^2}\sin(4s) + \frac{1}{s^2}\sin(s)$$

$$F_s(s) = \frac{5}{s^2}\sin(s) - \frac{1}{s}\cos(s) - \frac{1}{s^2}\sin 4s \; //$$

(11) Find the fourier sin transform of $e^{-|x|}$, and hence Show that $\int_0^{-\infty} \frac{x \sin mx}{1+x^2}\,dx = \frac{\pi}{2}e^{-m}$ for $m>0$.

$$F(x) = e^{-|x|}$$

WKT $|x| = \begin{cases} x, & x>0 \\ -x, & x<0 \end{cases}$

$$F(x) = \begin{cases} e^{-x}, & \text{for } x>0 \\ e^{x}, & \text{for } x<0 \end{cases}$$

Fourier sin transform of $F(x)$ is

$$F_s[F(x)] = f_s(s) = \int_0^{\infty} F(x)\sin(sx)\,dx$$

$$= \int_0^{\infty} e^{-x}\sin(sx)\,dx$$

$$F_s[F(x)] = \left\{\frac{e^{-x}}{(-1)^2+s^2}\left[-1\sin(sx) - s\cos(sx)\right]\right\}_0^{\infty}$$

$$= \frac{-1}{1+s^2}\left[e^{-x}\sin(sx) + s\cos(sx)\right]_0^{\infty}$$

$$= \frac{-1}{1+s^2}\left\{0 - 1\left[0+s(1)\right]\right\}$$

$$= \frac{-1}{1+s^2}(-s)$$

$\Rightarrow f_s(s) = \frac{s}{s^2+1}$

WKT inverse fourier sin transform.

$$F^{-1}[f_s(s)] = \frac{2}{\pi}\int_0^\infty f_s(s)\sin(sx)\,ds = F(x)$$

$$= \int_0^\infty \frac{s}{s^2+1}\sin(sx)\,ds = \frac{\pi}{2}e^{-x} \quad (1)$$

Let x=m > 0

$$(1) \Rightarrow \int_0^\infty \frac{s\sin(ms)}{1+s^2}\,ds = \frac{\pi}{2}e^{-m}$$

Let $s = x$

$$(2) \Rightarrow \int_0^\infty \frac{x\sin(mx)}{1+x^2}\,dx = \frac{\pi}{2}e^{-m}$$

(12) Find the Fourier cosin transform of $e^{-2x}+4e^{-3x}$

Let $F(x) = e^{-2x}+4e^{-3x}$

$$F_c[F(x)] = \int_0^\infty F(x)\cos(sx)\,dx$$

$$= \int_0^\infty (e^{-2x}+4e^{-3x})\cos(sx)\,dx$$

$$= \int_0^\infty e^{-2x}\cos(sx)\,dx + 4\int_0^\infty e^{-3x}\cos(sx)\,dx.$$

$$= \int_0^\infty e^{-2x}\cos(sx)\,dx = \frac{e^{-2x}}{(-2)^2+s^2}\left[-2\cos(sx)+s\sin(sx)\right]_0^\infty$$

$$= \frac{e^{-2x}}{s^2+4}\left[-2\cos(sx)+s\sin(sx)\right]_0^\infty$$

$$= \{0\} - \left\{\frac{1}{s^2+4}(-2)\right\}$$

$$= \frac{2}{s^2+4}$$

$$\int_0^\infty e^{-3x}\cos(sx)\,dx = \frac{e^{-3x}}{s^2+9}\left[-3\cos(sx)+s\sin(sx)\right]_0^\infty$$

$$= \frac{e^{-3x}}{s^2+9}\left[-3\cos(3x)+s\sin(sx)\right]_0^\infty$$

$$= [0] - \left\{\frac{-3}{s^2+9}\right\}$$

$$= \frac{3}{s^2+9}$$

$$f(s) = \frac{2}{s^2+4} + 4\,\frac{3}{s^2+9}$$

$$f_c(s) = \frac{2}{s^2+4} + \frac{12}{s^2+9}$$

(13) Find the Fourier cosin and sin transform of $2e^{-3x}+3e^{-2x}$

$$F(x) = 2e^{-3x}+3e^{-2x}$$

① $$F_c[F(x)] = \int_0^\infty F(x)\cos(sx)\,dx$$

$$= \int_0^\infty (2e^{-3x}+3e^{-2x})\cos(sx)\,dx$$

$$F_c[f(x)] = \int_0^\infty 2e^{-3x}\cos(sx)\,dx + \int_0^\infty 3e^{-2x}\cos(sx)\,dx \quad -①$$

$$\Rightarrow 2\int_0^\infty e^{-3x}\cos(sx)\,dx = \frac{e^{-3x}}{(-3)^2+s^2}\left[-3\cos(sx)+s\sin(sx)\right]_0^\infty$$

$$= \frac{e^{-3x}}{s^2+9}\left[-3\cos(sx)+s\sin(sx)\right]_0^\infty$$

$$= 0 - \left\{\frac{1}{s^2+9}(-3)\right\}$$

$$= 2\,\frac{3}{s^2+9}$$

$$= \frac{6}{s^2+9}$$

$$3\int_0^\infty e^{-2x}\cos(sx)\,dx = \frac{e^{-2x}}{s^2+4}\left[-2\cos(sx)+s\sin(sx)\right]_0^\infty$$

$$= \frac{e^{-2x}}{s^2+4}\left[-2\cos(sx) + s\sin(sx)\right]_0^\infty$$

$$= 0 - \left\{\frac{-2}{s^2+4}\right\}$$

$$= 3 \cdot \frac{2}{s^2+4}$$

$$= \frac{6}{s^2+4}$$

$$f_c(s) = \frac{6}{s^2+9} + \frac{6}{s^2+4} \quad //$$

(2) WKT

$$f_s[F(x)] = \int_0^\infty F(x)\sin(sx)\,dx$$

$$= \int_0^\infty (2e^{-3x} + 3e^{-2x})\sin(sx)\,dx$$

$$= 2\int_0^\infty e^{-3x}\sin(sx)\,dx + 3\int e^{-2x}\sin(sx)\,dx \quad -(1)$$

$$f_s(s) = 2\int_0^\infty e^{-3x}\sin(sx)\,dx = 2 \cdot \frac{e^{-3x}}{s^2+9}\left[-3\sin(sx) - s\cos(sx)\right]_0^\infty$$

$$= 0 - \frac{1}{s^2+9}\{0 - s\}$$

$$= \frac{2s}{s^2+9}$$

$$= 3\int_0^\infty e^{-2x}\sin(sx)\,dx = \frac{e^{-2x}}{s^2+4}\left[-2\sin(sx) - s\cos(sx)\right]_0^\infty$$

$$= 0 - \frac{1}{s^2+4}[0 - s]$$

$$= \frac{s}{s^2+4}$$

$$= \frac{3s}{s^2+4}$$

$$f_s(s) = \frac{2s}{s^2+9} + \frac{3s}{s^2+4} \quad //$$

(14) Find the Fourier cosin and sin transform of e^{-ax}

Let $F(x) = e^{-ax}$

WKT

① $Fc[F(x)] = \int_0^{\infty} F(x) \cos(sx)\, dx$

$= \int_0^{\infty} e^{-ax} \cos(sx)\, dx$

$= \frac{e^{-ax}}{(-a)^2+s^2} \left\{ [-a\cos(sx) + s\sin(sx)] \right\}_0^{\infty}$

$= \frac{e^{-ax}}{s^2+a^2} [-a\cos(sx) + s\sin(sx)]_0^{\infty}$

$= 0 - \frac{1}{s^2+a^2}(-a)$

$f_c(s) = \frac{a}{s^2+a^2}$

② WKT

$Fs[F(x)] = \int_0^{\infty} F(x) \sin(sx)\, dx$

$= \int_0^{\infty} e^{-ax} \sin(sx)\, dx$

$= \frac{e^{-ax}}{(-a)^2+s^2} [-a\sin(sx) - s\cos(sx)]_0^{\infty}$

$= \frac{e^{-ax}}{s^2+a^2} \{-a\sin(sx - s\cos(sx)\}_0^{\infty}$

$= 0 - \frac{1}{s^2+a^2} \{0 - s\}$

$f_s(s) = \frac{s}{s^2+a^2}$

(15) Find the fourier transform of sin & cosin of

$$F(x) = \begin{cases} x, & 0 < x < 2 \\ 0, & x \geq 2 \end{cases}$$

Given $F(x) = \begin{cases} x, & 0 < x < 2 \\ 0, & x \geq 2 \end{cases}$

WKT

① $F_S[F(x)] = \int_0^\infty F(x)\sin(sx)\,dx$

$= \int_0^2 F(x)\sin(sx)\,dx + \int_2^\infty F(x)\sin(sx)\,dx$

$= \int_0^2 x\sin(sx)\,dx$

$f_S(s) = x\int_0^2 \sin(sx)\,dx - \int_0^2 [1\,[\sin sx\,dx]]\,dx$

$= -\left[\frac{x\cos(sx)}{s}\right]_0^2 + \frac{1}{s}\int_0^2 \sin(sx)\,dx$

$= -\left[\frac{2\cos 2s - 0}{s}\right] + \frac{1}{s^2}[\sin(sx)]_0^2$

$= -\frac{2\cos 2s}{s} + \frac{1}{s^2}[\sin 2s - 0]$

$= \frac{1}{s^2}\sin 2s - \frac{2}{s}\cos 2s$

② $F_C[F(x)] = \int_0^\infty F(x)\cos(sx)\,dx$

$F(x) = \int_0^2 F(x)\cos(sx)\,dx + \int_2^\infty F(x)\cos(sx)\,dx$

$= \int_0^2 x\cos(sx)\,dx$

$= x\int_0^2 \cos(sx)\,dx - \int_0^2 [1\int\cos(sx)\,dx]\,dx$

$= \left[\frac{x\sin sx}{x}\right]_0^2 + \frac{1}{s^2}[\cos(sx)]_0^2$

$= \frac{2\sin 2s}{s} - 0 + \frac{1}{s^2}[\cos 2s - 1]$

$f_c(s) = \frac{1}{s^2}\cos 2s + \frac{2}{s}\sin 2s - \frac{1}{s^2}$ //

Z - transforms and Difference Equations:-

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + \dots$$

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 + \dots$$

$$(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^4 + \dots$$

$$(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \dots$$

Definition:-

Suppose $f(n)$ be a function in the variable n, such that the z-transform of $f(n)$ can be defined as

$Z[f(n)] = \sum_{n=0}^{\infty} f(n) z^{-n} = F(z)$, where Z is called the Z-transform operator.

Some important results:-

1) $f(n) = 1$

$$Z[1] = \sum_{n=0}^{\infty} 1 \cdot z^{-n}$$

$$= \sum_{n=0}^{\infty} \frac{1}{z^n}$$

$$= \frac{1}{z^0} + \frac{1}{z^1} + \frac{1}{z^2} + \frac{1}{z^3} + \dots$$

$$= 1 + \left(\frac{1}{z}\right) + \left(\frac{1}{z}\right)^2 + \left(\frac{1}{z}\right)^3 + \dots$$

$$= \left(1 - \frac{1}{z}\right)^{-1}$$

$$= \left(\frac{z-1}{z}\right)^{-1}$$

$$\boxed{Z[1] = \frac{z}{z-1}}$$

2) $f(n) = a^n$

$$\therefore Z[a^n] = \sum_{n=0}^{\infty} a^n \cdot z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n$$

$$= \left(\frac{a}{z}\right)^0 + \left(\frac{a}{z}\right)^1 + \left(\frac{a}{z}\right)^2 + \cdots$$

$$= 1 + \left(\frac{a}{z}\right) + \left(\frac{a}{z}\right)^2 + \cdots$$

$$= \left(1 - \frac{a}{z}\right)^{-1}$$

$$= \left(\frac{z-a}{z}\right)^{-1}$$

$$\boxed{z[a^n] = \frac{z}{z-a}}$$

for $a = -1$

$$z[(-1)^n] = \frac{z}{z+1}$$

③ $f(n) = n$

$$z[n] = \sum_{n=0}^{\infty} n \cdot z^{-n}$$

$$= 0 \cdot z^0 + 1 \cdot z^{-1} + 2 \cdot z^{-2} + 3 \cdot z^{-3} + 4 \cdot z^{-4} + \cdots$$

$$= 1\left(\frac{1}{z}\right) + 2\left(\frac{1}{z}\right)^2 + 3\left(\frac{1}{z}\right)^3 + 4\left(\frac{1}{z}\right)^4 + \cdots$$

$$= \frac{1}{z}\left[1 + 2\left(\frac{1}{z}\right) + 3\left(\frac{1}{z}\right)^2 + 4\left(\frac{1}{z}\right)^3 + \cdots\right]$$

$$= \frac{1}{z}\left[1 - \frac{1}{z}\right]^{-2}$$

$$= \frac{1}{z}\left[\frac{z-1}{z}\right]^{2}$$

$$= \frac{1}{z} \frac{z^2}{(z-1)^2}$$

$$\boxed{z[n] = \frac{z}{(z-1)^2}}$$

④ $f(n) = n^2$

WKT $z[n^p] = -z \frac{d}{dz} z[n^{p-1}]$ — ①

$\forall \; p = 2, 3, 4 \cdots$

Let $p = 2$

$$① \Rightarrow z[n^2] = -z \frac{d}{dz} z[n]$$

$$= -z \frac{d}{dz}\left[\frac{z}{(z-1)^2}\right]$$

$$= -z\left[\frac{(z-1)^2(1) - z\,2(z-1)(1)}{(z-1)^4}\right]$$

$$= -z\left[\frac{-z-1}{(z-1)^3}\right]$$

$$= \frac{z(z+1)}{(z-1)^3}$$

$$\boxed{z[n^2] = \frac{z^2+z}{(z-1)^3}}$$

① Find the z transform of $\cos n\theta$ and $\sin n\theta$.

Let $f(n) = e^{in\theta}$

$f(n) = \cos n\theta + i \sin n\theta$

$z[f(n)] = z[\cos n\theta] + i z[\sin n\theta]$ — ①

$z[f(n)] = z[e^{in\theta}]$

$z[f(n)] = z[[e^{i\theta}]^n]$

$$= \frac{z}{z-e^{i\theta}}$$

$$= \frac{z}{z-e^{i\theta}} \times \frac{z-e^{-i\theta}}{z-e^{-i\theta}}$$

$$= \frac{z[z-e^{-i\theta}]}{z^2 - ze^{-i\theta} - ze^{i\theta} + 1}$$

$$= \frac{z^2 - z[\cos\theta - i\sin\theta]}{z^2 - z(2\cos\theta) + 1}$$

$$= \frac{z^2 - z(\cos\theta) + i z \sin\theta}{z^2 - 2z\cos\theta + 1}$$

$$= \frac{z^2 - z\cos\theta}{z^2 - 2z\cos\theta + 1} + i\,\frac{z\sin\theta}{z^2 - 2z\cos\theta + 1}$$

$\therefore z[\cos n\theta] = \dfrac{z^2 - z\cos\theta}{z^2 - 2z\cos\theta + 1}$ $\quad\therefore z[\sin n\theta] = \dfrac{z\sin\theta}{z^2 - 2z\cos\theta + 1}$

② Find the z transform of $\cosh n\theta$ and $\sinh n\theta$

WKT $\cosh\theta = \frac{e^{\theta}+e^{-\theta}}{2}$

$$\cosh(n\theta) = \frac{e^{n\theta}+e^{-n\theta}}{2}$$

$$z[\cosh(n\theta)] = \frac{1}{2} z[e^{n\theta}+e^{-n\theta}]$$

$$= \frac{1}{2}\{z[e^{n\theta}]+z[e^{-n\theta}]\}$$

$$= \frac{1}{2}\{z[e^{\theta}]^n+z[e^{-\theta}]^n\}$$

$$= \frac{1}{2}\left\{\frac{z}{z-e^{\theta}}+\frac{1}{z-e^{-\theta}}\right\}$$

$$= \frac{z}{2}\left[\frac{1}{z-e^{\theta}}+\frac{1}{z-e^{-\theta}}\right]$$

$$= \frac{z}{2}\left[\frac{z-e^{-\theta}+z-e^{\theta}}{(z-e^{\theta})(z-e^{-\theta})}\right]$$

$$= \frac{z}{2}\left[\frac{2z-(e^{\theta}+e^{-\theta})}{z^2-z(e^{\theta}+e^{-\theta})}\right]$$

$$= \frac{z}{2}\left[\frac{2z-2\cosh\theta}{z^2-2z\cosh\theta+1}\right]$$

$$= \frac{z[z-\cosh\theta]}{z^2-2z\cosh\theta+1}$$

$$z[\cosh n\theta] = \frac{z^2-z\cosh\theta}{z^2-2z\cosh\theta+1}$$

WKT $\sinh\theta = \frac{e^{\theta}-e^{-\theta}}{2} \Rightarrow \sinh(n\theta) = \frac{e^{n\theta}-e^{-n\theta}}{2}$

$$z[\sinh(n\theta)] = \frac{1}{2} z[e^{n\theta}-e^{-n\theta}]$$

$$= \frac{1}{2}\{z[e^{n\theta}]-z[e^{-n\theta}]\}$$

$$= \frac{1}{2}\{z[e^{\theta}]^n-z[e^{-\theta}]^n\}$$

$$= \frac{1}{2}\left[\frac{z}{z-e^{\theta}}-\frac{z}{z-e^{-\theta}}\right]$$

$$= \frac{z}{2}\left[\frac{1}{z-e^{\theta}} - \frac{1}{z-e^{-\theta}}\right]$$

$$= \frac{z}{2}\left[\frac{z-e^{-\theta}-z+e^{\theta}}{(z-e^{\theta})(z-e^{-\theta})}\right]$$

$$= \frac{z}{2}\left[\frac{e^{\theta}-e^{-\theta}}{z^2-2z\cosh\theta+1}\right]$$

$$= \frac{z}{2}\left[\frac{2\sinh\theta}{z^2-2z\cosh\theta+1}\right]$$

$$z[\sinh n\theta] = \frac{z\sinh\theta}{z^2-2z\cosh\theta+1} \quad //$$

$$z[f(n)] = F(z)$$

(i) $z[a^n f(n)] = F\left(\frac{z}{a}\right)$

(ii) $z[a^{-n} f(n)] = F(az)$

(iii) $z[a^n n] = \frac{az}{(z-a)^2}$ called the daming room.

Find the z transform of the following functions:-

① $\cos\left(\frac{n\pi}{2} + \frac{\pi}{4}\right)$

Let $f(n) = \cos\left(\frac{n\pi}{2} + \frac{\pi}{4}\right)$

$$f(n) = \cos\left(\frac{n\pi}{2}\right)\cos\left(\frac{\pi}{4}\right) - \sin\left(\frac{n\pi}{2}\right)\sin\left(\frac{\pi}{4}\right)$$

$$= \frac{1}{\sqrt{2}}\cos\left(\frac{n\pi}{2}\right) - \frac{1}{\sqrt{2}}\sin\left(\frac{n\pi}{2}\right)$$

$$z[f(n)] = \frac{1}{\sqrt{2}}\, z\left[\cos\left(\frac{n\pi}{2}\right)\right] - \frac{1}{\sqrt{2}}\, z\left[\sin\left(\frac{n\pi}{2}\right)\right]$$

WKT $z[\cos n\theta] = \frac{z^2 - z\cos\theta}{z^2-2z\cos\theta+1}$

$$z\left[\cos\left(\frac{n\pi}{2}\right)\right] = \frac{z^2 - z\cos(\pi/2)}{z^2-2z\cos\left(\frac{\pi}{2}\right)+1}$$

$$Z\left[\cos\left(\frac{n\pi}{2}\right)\right] = \frac{z^2 - 0}{z^2 - 2z(0) + 1}$$

$$= \frac{z^2}{z^2+1}$$

$$Z[\sin(n\theta)] = \frac{z\sin\theta}{z^2 - 2z\cos\theta + 1}$$

$$Z\left[\sin\left(\frac{n\pi}{2}\right)\right] = \frac{z\sin(\pi/2)}{z^2 - 2z\cos\left(\frac{\pi}{2}\right) + 1}$$

$$= \frac{z}{z^2 - 0 + 1}$$

$$Z\left[\sin\left(\frac{n\pi}{2}\right)\right] = \frac{z}{z^2+1}$$

$$① \Rightarrow F(z) = \frac{1}{\sqrt{2}} \frac{z^2}{z^2+1} - \frac{1}{\sqrt{2}} \frac{z}{z^2+1}$$

$$F(z) = \frac{z^2 - z}{\sqrt{2}(z^2+1)} \quad //$$

② $\cos\left(\frac{n\pi}{2} + \theta\right)$

Let $f(n) = \cos\left(\frac{n\pi}{2} + \theta\right)$

$f(n) = \cos\left(\frac{n\pi}{2}\right)\cos\theta - \sin\left(\frac{n\pi}{2}\right)\sin\theta$

$Z[f(n)] = \cos\theta \, Z\left[\cos\left(\frac{n\pi}{2}\right)\right] - \sin\theta \, Z\left[\sin\left(\frac{n\pi}{2}\right)\right]$

WKT

$$Z[\cos n\theta] = \frac{z^2 - z\cos\theta}{z^2 - 2z\cos\theta + 1}$$

$$Z\left[\cos\left(\frac{n\pi}{2}\right)\right] = \frac{z^2 - z\cos(\pi/2)}{z^2 - 2z\cos(\pi/2) + 1}$$

$$Z\left[\cos\left(\frac{n\pi}{2}\right)\right] = \frac{z^2}{z^2+1}$$

$$Z[\sin(n\theta)] = \frac{z\sin\theta}{z^2-2z\cos\theta+1}$$

$$Z\left[\sin\left(\frac{n\pi}{2}\right)\right] = \frac{z\sin(\pi/2)}{z^2-2z\cos(\pi/2)+1}$$

$$= \frac{z}{z^2+1}$$

$$① \quad F(z) = \frac{z^2\cos\theta}{z^2+1} - \frac{z\sin\theta}{z^2+1}$$

$$F(z) = \frac{z^2\cos\theta - z\sin\theta}{z^2+1} \quad //$$

③ $2n + \sin\left(\frac{n\pi}{2}\right)+1$

Let $f(n) = 2n + \sin\left(\frac{n\pi}{2}\right)+1$

$$Z[f(n)] = 2Z[n] = Z\left[\sin\left(\frac{n\pi}{2}\right)\right] + Z[1] \quad - ①$$

$$Z(n) = \frac{z}{(z-1)^2} \Rightarrow \frac{2z}{(z-1)^2}$$

$$Z[\sin n\theta] = \frac{z\sin\theta}{z^2-2z\cos\theta+1}$$

$$Z\left[\sin\left(\frac{n\pi}{2}\right)\right] = \frac{z\sin(\pi/2)}{z^2-2z\cos(\pi/2)+1}$$

$$= \frac{z}{z^2+1}$$

$$Z[1] = \frac{z}{z-1}$$

$$Z[f(n)] = \frac{2z}{(z-1)^2} + \frac{z}{z^2+1} + \frac{z}{z-1} \quad //$$

④ $2n + \sin\left(\frac{n\pi}{4}\right)+1$

Let $f(n) = 2n + \sin\left(\frac{n\pi}{4}\right) + 1$

$$z[f(n)] = 2z[n] + z\left[\sin\left(\frac{n\pi}{4}\right)\right] + z[1] \quad -(1)$$

WKT $z[n] = \frac{z}{(z-1)^2}$

$$z[\sin n\theta] = \frac{z\sin\theta}{z^2 - 2z\cos\theta + 1}$$

$$z\left[\sin\left(\frac{n\pi}{4}\right)\right] = \frac{z\sin(\pi/4)}{z^2 - 2z\cos(\pi/4) + 1}$$

$$= \frac{z(1/\sqrt{2})}{z^2 - 2z\left(\frac{1}{\sqrt{2}}\right) + 1}$$

$$= \frac{\frac{z}{\sqrt{2}}}{\frac{\sqrt{2}z^2 - 2z + \sqrt{2}}{\sqrt{2}}}$$

$$z\left[\sin\left(\frac{n\pi}{4}\right)\right] = \frac{z}{\sqrt{2}z^2 - 2z + \sqrt{2}}$$

$$z[1] = \frac{z}{z-1}$$

$$(1) \Rightarrow F(z) = \frac{2z}{(z-1)^2} + \frac{z}{\sqrt{2}z^2 - 2z + \sqrt{2}} + \frac{z}{z-1}$$

(5) $3^n \cos\left(\frac{n\pi}{4}\right)$

Let $f(n) = \cos\left(\frac{n\pi}{4}\right)$

$$z[f(n)] = z\left[\cos\left(\frac{n\pi}{4}\right)\right]$$

$$F(z) = \frac{z^2 - z\cos(\pi/4)}{z^2 - 2z\cos(\pi/4) + 1}$$

$$= \frac{z^2 - z\left(\frac{1}{\sqrt{2}}\right)}{z^2 - 2z\left(\frac{1}{\sqrt{2}}\right) + 1}$$

$$F(z) = \frac{\sqrt{2}z^2 - z}{\sqrt{2}z^2 - 2z + \sqrt{2}}$$

WKT $z[a^n f(n)] = F\left(\frac{z}{a}\right)$

$z[3^n f(n)] = F\left(\frac{z}{3}\right)$

$$z\left[3^n \cos\left(\frac{n\pi}{4}\right)\right] = \frac{\sqrt{2}\left(\frac{z}{3}\right)^2 - \left(\frac{z}{3}\right)}{\sqrt{2}\left(\frac{z}{3}\right)^2 - 2\left(\frac{z}{3}\right) + \sqrt{2}}$$

$$= \frac{\frac{\sqrt{2}}{9} z^2 - \frac{z}{3}}{\frac{\sqrt{2}}{9} z^2 - \frac{2z}{3} + \sqrt{2}}$$

$$= \frac{\cancel{\frac{1}{9}}(\sqrt{2} z^2 - 3z)}{\cancel{\frac{1}{9}}(\sqrt{2} z^2 - 6z + 9\sqrt{2})}$$

$$= \frac{\sqrt{2} z^2 - 3z}{\sqrt{2} z^2 - 6z + 9\sqrt{2}} \quad //$$

6 $\sin(3n+5)$

Let $f(n) = \sin(3n+5)$

$f(n) = \sin(3n)\cos 5 + \cos(3n)\sin 5$

$z[f(n)] = \cos 5 \; z[\sin(3n)] + \sin 5 \; z[\cos(3n)]$ — ①

$$z[\cos(n\theta)] = \frac{z^2 - z\cos\theta}{z^2 - 2z\cos\theta + 1}$$

$$z[\cos 3n] = \frac{z^2 - z\cos 3}{z^2 - 2z\cos 3 + 1}$$

$$z[\sin n\theta] = \frac{z\sin\theta}{z^2 - 2z\cos\theta + 1}$$

$$z[\sin 3] = \frac{z\sin 3}{z^2 - 2z\cos 3 + 1}$$

$$F(z) = \frac{z\sin 3\cos 5}{z^2 - 2z\cos 3 + 1} + \frac{(z^2 - z\cos 3)\sin 3}{z^2 - 2z\cos 3 + 1}$$

$$= \frac{z\sin 3\cos 5 + z^2\sin 3 - z\sin 5\cos 3}{z^2 - 2z\cos 3 + 1}$$

$$= \frac{z[\sin 3\cos 5 - \sin 5\cos 3] + z^2\sin 3}{z^2 - 2z\cos 3 + 1}$$

$$= \frac{z\sin(-2) + z^2\sin 3}{z^2 - 2z\cos 3 + 1}$$

$$= \frac{z^2\sin 3 - z\sin 2}{z^2 - 2z\cos 3 + 1} \quad //$$

(7) $a^n \sin n\theta$

Let $f(n) = \sin n\theta$

$$Z[f(n)] = Z[\sin n\theta]$$

$$Z[\sin n\theta] = \frac{z\sin\theta}{z^2 - 2z\cos\theta + 1}$$

WKT $Z[a^n f(n)] = F\left(\frac{z}{a}\right)$

$$Z[a^n \sin n\theta] = \frac{\frac{z}{a}\sin n\theta}{\left(\frac{z}{a}\right)^2 - 2\left(\frac{z}{a}\right)\cos\theta + 1}$$

$$= \frac{\frac{z}{a}\sin n\theta}{\frac{z^2 - 2az\cos\theta + a^2}{a^2}}$$

$$= \frac{az\sin n\theta}{z^2 - 2az\cos\theta + a^2} \quad //$$

8) $a^{-n}\cos n\theta$

$$z[f(n)] = z[\cos n\theta]$$

$$F(z) = \frac{z^2 - z\cos\theta}{z^2 - 2z\cos\theta + 1}$$

WKT $z[a^{-n} f(n)] = F[az]$

$$z[a^{-n}\cos n\theta] = \frac{a^2z^2 - az\cos\theta}{a^2z^2 - 2az\cos\theta + 1} \quad //$$

Inverse z-transforms:-

Suppose $F(z)$ be a z-transform of $f(x)$, then the inverse z-transform of $F(z)$ can be defined as

$$z^{-1}[F(z)] = f(n).$$

Standard Results:-

1. $z^{-1}\left[F\left(\frac{z}{a}\right)\right] = a^n f(n)$
2. $z^{-1}[F(az)] = a^{-n} f(n)$
3. $z^{-1}\left[\frac{z}{z-1}\right] = 1$
4. $z^{-1}\left[\frac{z}{z-a}\right] = a^n$
5. $z^{-1}\left[\frac{z}{(z-1)^2}\right] = n$
6. $z^{-1}\left[\frac{z^2+z}{(z-1)^3}\right] = n^2$
7. $z^{-1}\left[\frac{az}{(z-a)^2}\right] = a^n \cdot n$
8. $z^{-1}\left[\frac{z}{z+1}\right] = (-1)^n$
9. $z^{-1}\left[\frac{z^2}{z^2+1}\right] = \cos\left(\frac{n\pi}{2}\right)$
10. $z^{-1}\left[\frac{z}{z^2+1}\right] = \sin\left(\frac{n\pi}{2}\right)$

Find the inverse z-transform of the following :-

① $\frac{z}{(z-2)(z-3)}$

Let $F(z) = \frac{z}{(z-2)(z-3)}$

$$\frac{F(z)}{z} = \frac{1}{(z-2)(z-3)}$$

$$\frac{1}{(z-2)(z-3)} = \frac{A}{(z-2)} + \frac{B}{(z-3)}$$

$$1 = A(z-3) + B(z-2) \quad -②$$

When $z = 2$

② $\Rightarrow 1 = A(2-3)$

$1 = -A$

$\boxed{A = -1}$

When $z = 3$

② $\Rightarrow 1 = B(3-2)$

$1 = B$

$\boxed{B = 1}$

$$① \Rightarrow \frac{F(z)}{z} = \frac{-1}{z-2} + \frac{1}{z-3}$$

$$F(z) = \frac{-z}{z-2} + \frac{z}{z-3}$$

$$Z^{-1}[F(z)] = -Z^{-1}\left[\frac{z}{z-2}\right] + Z^{-1}\left[\frac{z}{z-3}\right]$$

$$= -2^n + 3^n$$

$$f(n) = 3^n - 2^n$$

② $\frac{z}{z^2+7z+10}$

$$F(z) = \frac{z}{z^2+7z+10}$$

$$\frac{F(z)}{z} = \frac{1}{z^2+7z+10}$$

$$\frac{F(z)}{z} = \frac{1}{z^2+2z+5z+10}$$

$$\frac{F(z)}{z} = \frac{1}{(z+2)(z+5)}$$

$$\frac{F(z)}{z} = \frac{A}{z+2} + \frac{B}{z+5}$$

$$\frac{1}{(z+2)(z+5)} = A(z+5) + B(z+2) \quad -①$$

when $z=-2$

$\Rightarrow ① \Rightarrow 1 = A(-2+5)$

$1 = A(3)$

$A = \frac{1}{3}$

when $z=-5$

$1 = B(-5+2)$

$1 = B(-3)$

$B = -\frac{1}{3}$

$$\frac{F(z)}{z} = \frac{1}{3}\frac{1}{z+2} - \frac{1}{3}\frac{1}{z+5}$$

$$z^{-1}[F(z)] = \frac{1}{3}z^{-1}\left[\frac{z}{z-(-2)}\right] - \frac{1}{3}z^{-1}\left[\frac{z}{z-(-5)}\right]$$

$$f(n) = \frac{1}{3}(-2)^n - \frac{1}{3}(-5)^n$$

$$f(n) = \frac{1}{3}\left[(-2)^n - (-5)^n\right] //$$

③ $\frac{3z^2+2z}{(5z-1)(5z+2)}$

Let $F(z) = \frac{3z^2+2z}{(5z-1)(5z+2)}$

$$\frac{F(z)}{z} = \frac{3z+2}{(5z-1)(5z+2)} \quad -①$$

$$\frac{3z+2}{(5z-1)(5z+2)} = \frac{A}{5z-1} + \frac{B}{5z+2}$$

$$3z+2 = A(5z+2) + B(5z-1) \quad -②$$

When $z=\frac{1}{5}$

$② \Rightarrow \frac{3}{5} + 2 = A(3)$

$\Rightarrow \frac{13}{5} = 3A \Rightarrow \boxed{A = \frac{13}{15}}$

when $z = -\frac{2}{5}$

$② \Rightarrow 3\left(-\frac{2}{5}\right) + 2 = B(-2-1)$

$\Rightarrow -\frac{6}{5} + 2 = -3B$

$\Rightarrow \frac{-6+10}{5} = -3B \quad \boxed{B = \frac{-4}{15}}$

$$① \Rightarrow \frac{F(z)}{z} = \frac{13}{15}\frac{1}{(5z-1)} - \frac{4}{15}\cdot\frac{1}{(5z+2)}$$

$$= \frac{13}{15\times 5}\cdot\frac{z}{\left(z-\frac{1}{5}\right)} - \frac{4}{15\times 5}\frac{z}{\left(z+\frac{2}{5}\right)}$$

$$F(z) = \frac{13}{75}\left[\frac{z}{z-\frac{1}{5}}\right] - \frac{4}{75}\left[\frac{z}{z-\left(-\frac{2}{5}\right)}\right]$$

$$z^{-1}[F(z)] = \frac{13}{75}z^{-1}\left[\frac{z}{z-\frac{1}{5}}\right] - \frac{4}{75}z^{-1}\left[\frac{z}{z-\left(-\frac{2}{5}\right)}\right]$$

$$f(n) = \frac{13}{75}\left(\frac{1}{5}\right)^n - \frac{4}{75}\left[\frac{-2}{5}\right]^n //$$

④ $\frac{18z^2}{(2z-1)(4z+1)}$

Let $F(z) = \frac{18z^2}{(2z-1)(4z+1)}$

$$\frac{F(z)}{z} = \frac{18z}{(2z-1)(4z+1)} \quad -①$$

$$\frac{18z}{(2z-1)(4z+1)} = \frac{A}{(2z-1)} + \frac{B}{(4z+1)} \quad -②$$

$$18z = A(4z+1) + B(2z-1)$$

when $z = \frac{1}{2}$

$② \Rightarrow 9 = 3A \Rightarrow \boxed{A=3}$

when $z = -\frac{1}{4}$

$② \Rightarrow 18\left(-\frac{1}{4}\right) = B\left[2\left(-\frac{1}{2}\right)-1\right]$

$\Rightarrow \frac{-9}{2} = -\frac{3}{2}B$

$\boxed{B=3}$

$$① \Rightarrow F(z) = \frac{3}{(2z-1)} + \frac{3}{(4z+1)}$$

$$= \frac{3z}{(2z-1)} + \frac{3z}{(4z+1)}$$

$$= \frac{3}{2}\left[\frac{z}{z-\frac{1}{2}}\right] + \frac{3}{4}\left[\frac{z}{z+\frac{1}{4}}\right]$$

$$z^{-1}[F(z)] = \frac{3}{2}z^{-1}\left[\frac{z}{z-\frac{1}{2}}\right] + \frac{3}{4}z^{-1}\left[\frac{z}{z-\left(-\frac{1}{4}\right)}\right]$$

$$f(n) = \frac{3}{2}\left(\frac{1}{2}\right)^n + \frac{3}{4}\left(-\frac{1}{4}\right)^n //$$

⑤ $\frac{2z^2+3z}{(z+2)(z-4)}$

$$F(z) = \frac{2z^2+3z}{(z+2)(z-4)}$$

$$\frac{F(z)}{z} = \frac{2z+3}{(z+2)(z-4)} \quad -①$$

$$\frac{2z+3}{(z+2)(z-4)} = \frac{A}{(z+2)} + \frac{B}{(z-4)}$$

$$2z+3 = A(z-4) + B(z+2) \quad -②$$

when $z=4$

② $\Rightarrow 11 = B(6)$

$B = \frac{11}{6}$

when $z=-2$

$-1 = A(-6)$

$A = \frac{1}{6}$

① $\Rightarrow \frac{F(z)}{z} = \frac{1}{6}\frac{1}{(z+2)} + \frac{11}{6}\frac{1}{(z-4)}$

$$F(z) = \frac{1}{6}\frac{z}{(z+2)} + \frac{11}{6}\frac{z}{(z-4)}$$

$$z^{-1}[F(z)] = \frac{1}{6} z^{-1}\left[\frac{z}{z+2}\right] + \frac{11}{6} z^{-1}\left[\frac{z}{z-4}\right]$$

$$f(n) = \frac{1}{6}(-2)^n + \frac{11}{6}(4)^n \,//$$

⑥ $\frac{8z^2}{(2z-1)(4z-1)}$

$$F(z) = \frac{8z^2}{(2z-1)(4z-1)}$$

$$\frac{F(z)}{z} = \frac{8z}{(2z-1)(4z-1)} \quad -①$$

$$\frac{8z}{(2z-1)(4z-1)} = \frac{A}{(2z-1)} + \frac{B}{(4z-1)}$$

$$8z = A(4z-1) + B(2z-1) \quad -②$$

when $z = \frac{1}{4}$

② $\Rightarrow 2 = B\left(2\left(\frac{1}{4}\right)-1\right)$

$\Rightarrow 2 = B\left(\frac{1}{2}-1\right)$

$2 = B\left(\frac{1-2}{2}\right)$

$B = -4$

when $z = \frac{1}{2}$

② $\Rightarrow 4 = A(2-1)$

$\Rightarrow 4 = A$

$A = 4$

① $\Rightarrow \frac{F(z)}{z} = \frac{4}{(2z-1)} - \frac{4}{(4z-1)}$

$F(z) = 4\frac{z}{(2z-1)} - 4\cdot\frac{z}{(4z-1)}$

$z^{-1}[F(z)] = 4\, z^{-1}\left[\frac{z}{2z-1}\right] - 4\, z^{-1}\left[\frac{z}{4z-1}\right]$

$= \frac{\overset{2}{\cancel{4}}}{\cancel{2}} z^{-1}\left[\frac{z}{z-\frac{1}{2}}\right] - \frac{4}{4}\, z^{-1}\left[\frac{z}{z-\frac{1}{4}}\right]$

$f(n) = 2\left(\frac{1}{2}\right)^n - \left(\frac{1}{4}\right)^n$ //

⑦ $\frac{4z^2-2z}{z^3-5z^2+8z-4}$

$F(z) = \frac{4z^2-2z}{z^3-5z^2+8z-4}$

$\frac{F(z)}{z} = \frac{4z-2}{z^3-5z^2+8z-4}$

$= \frac{4z-2}{z^3-5z^2+8z-4}$

$= \frac{4z-2}{(z-1)(z-2)^2}$

$\frac{4z-2}{(z-1)(z-2)^2} = \frac{A}{z-1} + \frac{B}{z-2} + \frac{C}{(z-2)^2}$

$4z-2 = A(z-2)^2 + B(z-1)(z-2) + C(z-1)$ — ②

when $z=1$

$2 = A$

$\boxed{A=2}$

when $z = 2$

$6 = C$

$\boxed{C=6}$

$A+B=0$

$B=-A$

$\boxed{B=-2}$

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$$\textcircled{1} \Rightarrow \frac{F(z)}{z} = \frac{2}{z-1} - \frac{2}{z-2} + \frac{6}{(z-2)^2}$$

$$= 2\frac{z}{z-1} - 2\frac{z}{z-2} + 3\frac{2z}{(z-2)^2}$$

$$z^{-1}[F(z)] = 2z^{-1}\left[\frac{z}{z-1}\right] - 2z^{-1}\left[\frac{z}{z-2}\right] + 3z^{-1}\left[\frac{2z}{(z-2)^2}\right]$$

$$= 2(1) - 2(2^n) + 3.n2^n //$$

Difference Equations:-

Step ① :- Express the given difference equation in the notation of $y_n, y_{n+1}, y_{n+2} \ldots\ldots$

Step ② :- Apply z-transform on both sides and substitute

$$z[y_{n+2}] = z^2\left[\bar{y}(z) - y_0 - \frac{y_1}{z}\right]$$

$$z[y_{n+1}] = z\left[\bar{y}(z) - y_0\right]$$

$$z[y_n] = \bar{y}(z)$$

Step ③ :- write $\bar{y}(z)$ has a function of z, hence apply the inverse z-transform and find $y(n)$.

① Solve the difference equation using z-transform $y_{n+2} - 4y_n = 0$, Subject to the conditions $y_0 = 0, y_1 = 2$.

Given $y_{n+2} - 4y_n = 0 \quad y_0 = 0, \; y_1 = 2$

$$\Rightarrow z[y_{n+2}] - 4z[y_n] = z[0]$$

$$\Rightarrow z^2\left[\bar{y}(z) - y_0 - \frac{y_1}{z}\right] - 4\bar{y}(z) = 0$$

$$\Rightarrow z^2\left[\bar{y}(z) - 0 - \frac{2}{z}\right] - 4\bar{y}(z) = 0$$

$$\Rightarrow z^2\bar{y}(z) - 2z - 4\bar{y}(z) = 0$$

$$(z^2 - 4)\bar{y}(z) = 2z$$

$$\Rightarrow \bar{y}(z) = \frac{2z}{(z^2-4)}$$

$$\bar{y}(z) = \frac{2z}{(z+2)(z-2)}$$

$$\frac{\bar{y}(z)}{z} = \frac{2}{(z-2)(z+2)} \quad -①$$

$$\frac{z}{(z-2)(z+2)} = \frac{A}{(z-2)} + \frac{B}{(z+2)}$$

$$2 = A(z+2) + B(z-2) - ②$$

when $z=2$

$② \Rightarrow 2 = 4A$

$A = \frac{1}{2}$

when $z=-2$

$② \Rightarrow 2 = -4B$

$B = -\frac{1}{2}$

$$① \Rightarrow \frac{\bar{y}(z)}{z} = \frac{1}{2}\cdot\frac{1}{z-2} - \frac{1}{2}\frac{1}{z+2}$$

$$\bar{y}(z) = \frac{1}{2}\frac{z}{z-2} - \frac{1}{2}\frac{z}{z+2}$$

$$z^{-1}[\bar{y}(z)] = \frac{1}{2} z^{-1}\left[\frac{z}{z-2}\right] - \frac{1}{2} z^{-1}\left[\frac{z}{z+2}\right]$$

$$y(n) = \frac{1}{2}(2)^n - \frac{1}{2}(-2)^n$$

② Using z-transform solve the difference equation $y_{n+2} + 6y_{n+1} + 9y_n = 2^n$, subject to the conditions $y_0=0$, $y_1=0$.

Given $y_{n+2} + 6y_{n+1} + 9y_n = 2^n$, $y_0=0$, $y_1=0$

$$z[y_{n+2}] + 6z[y_{n+1}] + 9z[y_n] = z[2^n]$$

$$z^2\left[\bar{y}(z) - y_0 - \frac{y_1}{z}\right] + 6z[\bar{y}(z) - y_0] + 9\bar{y}(z) = \frac{z}{z-2}$$

$$z^2\bar{y}(z) + 6z\bar{y}(z) + 9\bar{y}(z) = \frac{z}{z-2}$$

$$(z^2+6z+9)\bar{y}(z) = \frac{z}{z-2}$$

$$\bar{y}(z) = \frac{z}{(z-2)(z+3)^2}$$

$$\frac{\bar{Y}(z)}{z} = \frac{1}{(z-2)(z+3)^2} \quad -①$$

$$\frac{1}{(z-2)(z+3)^2} = \frac{A}{z-2} + \frac{B}{z+3} + \frac{C}{(z+3)^2}$$

$$1 = A(z+3)^2 + B(z-2)(z+3) + C(z-2) \quad -②$$

when $z = 2$

$② \Rightarrow 1 = A(2+3)^2$

$1 = 25A$

$A = \frac{1}{25}$

when $z = -3$

$② \Rightarrow 1 = C(-5)$

$C = \frac{-1}{5}$

when

$A + B = 0$

$B = -A$

$B = \frac{-1}{25}$

$$① \Rightarrow \frac{\bar{y}(z)}{z} = \frac{1}{25} \cdot \frac{1}{z-2} - \frac{1}{25} \frac{1}{z+3} - \frac{1}{5} \cdot \frac{1}{(z+3)^2}$$

$$\bar{y}(z) = \frac{1}{25} \frac{z}{z-2} - \frac{1}{25} \frac{z}{z+3} - \frac{1}{5} \cdot \frac{z}{(z+3)^2}$$

$$\bar{y}(z) = \frac{1}{25} \cdot \frac{z}{z-2} - \frac{1}{25} \cdot \frac{z}{z-(-3)} + \frac{1}{5} \frac{(-3z)}{(z-(-3))^2}$$

$$z^{-1}[\bar{y}(z)] = \frac{1}{25} z^{-1}\left[\frac{z}{z-2}\right] - \frac{1}{25} z^{-1}\left[\frac{z}{z-(-3)}\right] + \frac{1}{15} \frac{(-3z)}{(z-(-3))^2}$$

$$\Rightarrow y(n) = \frac{1}{25}(2)^n - \frac{1}{25}(-3)^n + \frac{1}{15}(-3)^n \cdot n \;//$$

③ Solve $u_{n+2} - 3u_{n+1} + 2u_n = 2^n$. Given $u_0 = 0$, $u_1 = 1$ by using z-transform.

Given $u_{n+2} - 3u_{n+1} + 2u_n = 2^n$, $u_0 = 0$, $u_1 = 1$

$$\Rightarrow z[u_{n+2}] - 3z[u_{n+1}] + 2z[u_n] = z[2^n]$$

$$\Rightarrow z^2\left[\bar{u}(z) - u_0 - \frac{u_1}{z}\right] - 3z[\bar{u}(z) - u_0] + 2\bar{u}(z) = \frac{z}{z-2}$$

$$\Rightarrow z^2\bar{u}(z) - z - 3z\bar{u}(z) + 2\bar{u}(z) = \frac{z}{z-2}$$

$$(z^2 - 3z + 2)\bar{u}(z) = \frac{z}{z-2} + z$$

$$(z-1)(z-2)\bar{u}(z) = \frac{z + z(z-2)}{z-2}$$

$$(z-1)(z-2)\bar{u}(z) = \frac{z^2 - z}{z-2}$$

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$$\bar{u}(z) = \frac{z^2 - z}{(z-1)(z-2)^2}$$

$$\bar{u}(z) = \frac{z(z-1)}{(z-1)(z-2)^2}$$

$$\bar{u}(z) = \frac{z}{(z-2)^2}$$

$$\bar{u}(z) = \frac{1}{2} \cdot \frac{2z}{(z-2)^2}$$

$$z^{-1}[\bar{u}(z)] = \frac{1}{2} z^{-1}\left[\frac{2z}{(z-2)^2}\right]$$

$$u(n) = \frac{1}{2} n 2^n$$

$$u(n) = 2^{n-1} n \quad //$$

(4) Solve the difference equation $u_{n+2} + 2u_{n+1} + u_n = n$, subject to the conditions $u_0 = 0$, $u_1 = 0$.

Given $u_{n+2} + 2u_{n+1} + u_n = n$, $u_0 = 0$, $u_1 = 0$

$$\Rightarrow z[u_{n+2}] + 2z[u_{n+1}] + z[u_n] = z[n]$$

$$\Rightarrow z^2\left[\bar{u}(z) - u_0 - \frac{u_1}{z}\right] + 2z[\bar{u}(z)] + \bar{u}(z) = \frac{z}{(z-1)^2}$$

$$z^2\bar{u}(z) + 2z\bar{u}(z) + \bar{u}(z) = \frac{z}{(z-1)^2}$$

$$(z^2 + 2z + 1)\bar{u}(z) = \frac{z}{(z-1)^2}$$

$$(z+1)^2 \bar{u}(z) = \frac{z}{(z-1)^2}$$

$$\bar{u}(z) = \frac{z}{(z-1)^2(z+1)^2}$$

$$\frac{\bar{u}(z)}{z} = \frac{1}{(z-1)^2(z+1)^2} \quad \text{— (1)}$$

$$\frac{1}{(z-1)^2(z+1)^2} = \frac{A}{z-1} + \frac{B}{(z-1)^2} + \frac{C}{(z+1)} + \frac{D}{(z+1)^2}$$

$$\Rightarrow 1 = A(z-1)(z+1)^2 + B(z+1)^2 + C(z+1)(z-1)^2 + D(z-1)^2 \quad \text{— (2)}$$

when $z=1$

② $\Rightarrow 1 = 4B$

$B = \frac{1}{4}$

when $z=-1$

② $\Rightarrow 1 = 4D$

$D = \frac{1}{4}$

$-A + B + C + D = 1$

$-A + \frac{1}{4} + C + \frac{1}{4} = 1$

$-A + C = \frac{1}{2}$ — ③

when $-A + C = 0$ — ④

③ + ④ $\Rightarrow 2C = \frac{1}{2}$

$C = \frac{1}{4}$

$A = -\frac{1}{4}$

① $\Rightarrow \frac{\bar{u}(z)}{z} = -\frac{1}{4}\frac{1}{z-1} + \frac{1}{4}\frac{1}{(z-1)^2} + \frac{1}{4}\cdot\frac{1}{(z+1)} + \frac{1}{4}\frac{1}{(z+1)^2}$

$\Rightarrow \bar{u}(z) = -\frac{1}{4}\frac{z}{z-1} + \frac{1}{4}\frac{z}{(z-1)^2} + \frac{1}{4}\frac{z}{(z+1)} + \frac{1}{4}\frac{z}{(z+1)^2}$

$z^{-1}[\bar{u}(z)] = -\frac{1}{4} z^{-1}\left[\frac{z}{z-1}\right] + \frac{1}{4} z^{-1}\left[\frac{z}{(z-1)^2}\right] + \frac{1}{4} z^{-1}\left[\frac{z}{z-(-1)}\right] + \frac{1}{4}\left[\frac{(-1)z}{z-(-1)^2}\right]$

$u(n) = -\frac{1}{4} + \frac{1}{4}n + \frac{1}{4}(-1)^n + \frac{1}{4}(-1)^n \cdot n$ //

⑤ Solve the difference equation $u_{n+2} + 4u_{n+1} + 3u_n = 3^n$, subject to the conditions $u_0 = 0$, $u_1 = 1$

Given $u_{n+2} + 4u_{n+1} + 3u_n = 3^n$, $u_0 = 0$, $u_1 = 1$

$\Rightarrow z[u_{n+2}] + 4z[u_{n+1}] + 3z[u_n] = z[3^n]$

$\Rightarrow z^2\left[\bar{u}(z) - u_0 - \frac{u_1}{z}\right] + 4z[\bar{u}(z) - u_0] + 3\bar{u}(z) = \frac{z}{z-3}$

$z^2\bar{u}(z) - z + 4z\bar{u}(z) + 3\bar{u}(z) = \frac{z}{z-3}$

$[z^2 + 4z + 3]\bar{u}(z) = \frac{z}{z-3} + z$

$(z+3)(z+1)\bar{u}(z) = \frac{z + z(z-3)}{z-3}$

$\bar{u}(z) = \frac{z + z^2 - 3z}{(z+1)(z-3)(z+3)}$

$\bar{u}(z) = \frac{z^2 - 2z}{(z+1)(z-3)(z+3)}$

$\frac{\bar{u}(z)}{z} = \frac{z-2}{(z+1)(z+3)(z-3)}$ — ①

$$\frac{z-2}{(z+1)(z+3)(z-3)} = \frac{A}{z+1} + \frac{B}{z+3} + \frac{C}{z-3}$$

$$z-2 = A(z+3)(z-3) + B(z+1)(z-3) + C(z+1)(z+3) \quad \text{—②}$$

when $z=-3$ $z=3$ $z=-1$

② $\Rightarrow -5 = B(-2)(-6)$ ② $\Rightarrow 1 = C(4)(6)$ ② $\Rightarrow -3 = A(2)(-4)$

$-5 = 12B$ $1 = 24C$ $-3 = -8A$

$B = \frac{-5}{12}$ $C = \frac{1}{24}$ $A = \frac{3}{8}$

$$① \Rightarrow \frac{\bar{u}(z)}{z} = \frac{3}{8}\frac{1}{z+1} - \frac{5}{12}\frac{1}{z+3} + \frac{1}{24}\frac{1}{z-3}$$

$$\bar{u}(z) = \frac{3}{8}\frac{z}{z+1} - \frac{5}{12}\cdot\frac{z}{z+3} + \frac{1}{24}\frac{z}{z-3}$$

$$z^{-1}[\bar{u}(z)] = \frac{3}{8}z^{-1}\left[\frac{z}{z+1}\right] - \frac{5}{12}z^{-1}\left[\frac{z}{z+3}\right] + \frac{1}{24}z^{-1}\left[\frac{z}{z-3}\right]$$

$$u(n) = \frac{3}{8}(-1)^n - \frac{5}{12}(-3)^n + \frac{1}{24}(3)^n \,//$$

⑥ Solve the difference equation $u_{n+2} - 3u_{n+1} + 2u_n = 3^n$, subject to the conditions $u_0 = u_1 = 0$.

Given $u_{n+2} - 3u_{n+1} + 2u_n = 3^n$, $u_0 = 0$, $u_1 = 0$

$$\Rightarrow z[u_{n+2}] - 3z[u_{n+1}] + 2z[u_n] = z[3^n]$$

$$\Rightarrow z^2\left[\bar{u}(z) - u_0 - \frac{u_1}{z}\right] - 3z[\bar{u}(z) - u_0] + 2\bar{u}(z) = \frac{z}{z-3}$$

$$z^2\bar{u}(z) - 3z\,\bar{u}(z) + 2\bar{u}(z) = \frac{z}{z-3}$$

$$[z^2 - 3z + 2]\,\bar{u}(z) = \frac{z}{z-3}$$

$$(z-1)(z-2)\,\bar{u}(z) = \frac{z}{z-3}$$

$$\bar{u}(z) = \frac{z}{(z-1)(z-2)(z-3)} \quad \text{—①}$$

$$\frac{\bar{u}(z)}{z} = \frac{1}{(z-1)(z-2)(z-3)}$$

$$\frac{1}{(z-1)(z-2)(z-3)} = \frac{A}{z-1} + \frac{B}{z-2} + \frac{C}{z-3}$$

$$1 = A(z-2)(z-3) + B(z-1)(z-3) + C(z-1)(z-2) \quad \text{--- (2)}$$

when $z=2$

(2) $\Rightarrow 1 = B(1)(1)$

$\boxed{B=1}$

$z = 1$

$1 = A(-1)(-2)$

$A = 2$

$z = 3$

$1 = C(2)(1)$

$C = \frac{1}{2}$

$$\frac{\bar{u}(z)}{z} = \frac{2}{(z-1)} + \frac{1}{(z-2)} + \frac{1}{2}\left(\frac{1}{z-3}\right)$$

$$\bar{u}(z) = 2\left(\frac{z}{z-1}\right) + \frac{z}{z-2} + \frac{1}{2}\left(\frac{z}{z-3}\right)$$

$$z^{-1}[\bar{u}(z)] = 2z^{-1}\left[\frac{z}{z-1}\right] + z^{-1}\left[\frac{z}{z-2}\right] + \frac{1}{2} z^{-1}\left[\frac{z}{z-3}\right]$$

$$u(n) = 2(1) + (2)^n + \frac{1}{2}(3^n) \quad //$$

Module - 2

FOURIER SERIES

Introduction :-

Expressing the function $f(x)$ in the combinations of constants and trignometric ratios in any other interval $(0,\pi)$ $(0,2\pi)$.... is called the Fourier series.

$$\therefore f(x) = \frac{a_0}{2} + a_1\cos x + a_2\cos 2x + a_3\cos 3x + \dots + a_n\cos nx + \dots + b_1\sin x + b_2\sin 2x + b_3\sin 3x + \dots + b_n\sin nx + \dots$$

$$\Rightarrow f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} b_n\sin nx$$

$$\Rightarrow f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n\cos nx + b_n\sin nx)$$

where a_0 is called the constant.

a_n, b_n are called fourier co-efficients.

Fourier series Expansion of $f(x)$ over the period 2π:-

The Fourier series expansion of $f(x)$ over the interval $(c, c+2\pi)$ can be denoted as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n\cos nx + b_n\sin nx) \quad \text{—} ①$$

where $a_0 = \frac{1}{\pi}\int_0^{c+2\pi} f(x)\,dx$

$$a_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\sin nx\,dx$$

This is also called direchlet's property.

If $c=0$, then the constant and the fourier coefficients of $f(x)$ can be evaluated as $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

If $c=-\pi$, then the fourier series coefficients of $f(x)$ in the interval $(-\pi,\pi)$ can be evaluated as

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

Note:-

① Suppose $f(x)$ is the continuous function in the interval $[0,2\pi]$ then $f(x)$ is an even function when $f(2\pi-x)=f(x)$ and $f(x)$ is an odd function when $f(2\pi-x)=-f(x)$

② If $f(x)$ is a discontinuous function over the interval $[0,2\pi]$ $f(x)=\begin{cases}\phi(x), & 0\le x\le\pi\\ \psi(x), & \pi\le x<2\pi\end{cases}$, then

$f(x)$ is an even function when $\phi(2\pi-x)=\psi(x)$ and $f(x)$ is an odd function when $\phi(2\pi-x)=-\psi(x)$

③ Suppose $f(x)$ is the continuous function in the interval $[-\pi,\pi]$, then

$f(x)$ is an even function when $f(-x)=f(x)$ and $f(x)$ is an odd function when $f(-x)=-f(x)$.

④ suppose $f(x)$ is a discontinuous function in the interval $[-\pi, \pi]$, that is $f(x) = \begin{cases} \phi(x), & -\pi \le x < 0 \\ \psi(x), & 0 \le x < \pi \end{cases}$, then

$f(x)$ is an even function when $\phi(-x) = \psi(x)$ and

$f(x)$ is an function when $\phi(-x) = \psi(x)$ and

$f(x)$ is an odd function when $\phi(-x) = -\psi(x)$.

Even and odd functions:-

WKT the fourier series expansion of $f(x)$ over the period 2π is $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n \cos nx + b_n \sin nx]$, then

1. If $f(x)$ is an even function, then $b_n = 0$, hence the fourier series of $f(x)$ can be expanded as $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$ is also called as half-range cosine series in the $[0, \pi]$ for which.

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x) \cos \pi x\,dx$$

2. If $f(x)$ is an odd function; then a_0 and a_n values will be zero, hence the fourier series expansion of $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$ is also called the fourier half-range sin series in the $[0, \pi]$ for which $b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx\,dx$

Important results:-

1. $\sin(n\pi) = 0$, $\sin(2n\pi) = 0$, $\forall n \in Z$
2. $\cos(n\pi) = (-1)^n$, $\cos(2n\pi) = 1$, $\forall n \in Z$
3. $\sin\left(n + \frac{1}{2}\right)\pi = (-1)^n$, $\forall n \in Z$
4. $\cos\left(n + \frac{1}{2}\right)\pi = 0$, $\forall n \in Z$

5. $\sin\left(\frac{n\pi}{2}\right) = \begin{cases} (-1)^{\frac{n-1}{2}}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases}$

6. $\cos\left(\frac{n\pi}{2}\right) = \begin{cases} 0, & \text{if } n \text{ is odd} \\ (-1)^{n/2}, & \text{if } n \text{ is even} \end{cases}$

① Obtain the fourier series of $f(x) = \frac{\pi - x}{2}$ in $[0, 2\pi]$ and hence deduce that $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$

Given $f(x) = \frac{\pi - x}{2}$, $x \in [0, 2\pi]$

$$f(2\pi - x) = \frac{\pi - (2\pi - x)}{2}$$

$$= \frac{\pi - 2\pi + x}{2}$$

$$= \frac{-\pi + x}{2}$$

$$f(2\pi - x) = -f(x)$$

$\therefore f(x)$ is an odd function in $[0, 2\pi]$

$a_0 = 0$, $a_n = 0$

The fourier series of

$$\Rightarrow f(x) = \sum_{n=1}^{\infty} b_n \sin nx \quad \text{—①}$$

when $b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx$

$$= \frac{2}{\pi} \int_0^{\pi} \left(\frac{\pi - x}{2}\right) \sin nx \, dx$$

$$= \frac{1}{\pi} \int_0^{\pi} (\pi - x) \sin nx \, dx$$

$$= \frac{1}{\pi} \left\{ (\pi - x) \int_0^{\pi} \sin nx \, dx - \int_0^{\pi} (-1) \left[\int \sin nx \, dx\right] dx \right\}$$

$$= \frac{1}{\pi} \left\{ -\frac{1}{n} (\pi - x) \cos nx \Big]_0^{\pi} - \frac{1}{n} \int_0^{\pi} \cos nx \, dx \right\}$$

$$= \frac{1}{\pi} \left\{ \frac{-1}{n} (\pi - x) \cos nx \Big]_0^{\pi} - \frac{1}{n^2} [\sin nx]_0^{\pi} \right\}$$

$$= \frac{1}{\pi} \left\{ -\frac{1}{n} [0-\pi] - \frac{1}{n^2} [0-0] \right\}$$

$$= \frac{1}{\pi} \left[\frac{\pi}{n} \right]$$

$$b_n = \frac{1}{n}$$

$$① \Rightarrow f(x) = \sum_{n=1}^{\infty} \frac{1}{n} \sin nx$$

$$\frac{\pi - x}{2} = \sum_{n=1}^{\infty} \frac{1}{n} \sin nx \quad — ②$$

$$② \Rightarrow \frac{\pi - \pi/2}{2} = \sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{n\pi}{2}\right)$$

$$\Rightarrow \frac{\pi}{4} = \sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{n\pi}{2}\right)$$

$$\Rightarrow \frac{\pi}{4} = \frac{1}{1} \sin\left(\frac{\pi}{2}\right) + \frac{1}{2} \sin(\pi) + \frac{1}{3} \sin\left(\frac{3\pi}{2}\right) + \frac{1}{4} \sin(2\pi) + \frac{1}{5} \sin\left[\frac{5\pi}{2}\right] + \cdots$$

$$\Rightarrow \frac{\pi}{4} = 1 + 0 - \frac{1}{3} + 0 + \frac{1}{5} - \frac{1}{7} + \cdots$$

$$\Rightarrow \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \;//$$

② Obtain a series of $f(x) = \left(\frac{\pi - x}{2}\right)^2$ in $[0, 2\pi]$, Hence deduce that 1. $\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$

2. $\frac{\pi^2}{12} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$

$$\Rightarrow f(x) = \left(\frac{\pi - x}{2}\right)^2, \quad x \in [0, 2\pi]$$

$$f(x) = \left(\frac{x - \pi}{2}\right)^2 = \frac{1}{4}(x-\pi)^2$$

$$f(2\pi - x) = \frac{1}{4}[2\pi - x - \pi]^2$$

$$= \frac{1}{4}[\pi - x]^2$$

$$= \frac{1}{4}[x - \pi]^2$$

$$f(2\pi - x) = f(x)$$

$f(x)$ is an even function.

$$\Rightarrow b_n = 0$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \text{—} ①$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} \frac{1}{4}(x-\pi)^2\,dx$$

$$= \frac{1}{2\pi}\int_0^{\pi}(x^2 - 2\pi x + \pi^2)\,dx$$

$$= \frac{1}{2\pi}\left[\frac{x^3}{3} - \frac{2\pi x^2}{2} + \pi^2 x\right]_0^{\pi}$$

$$= \frac{1}{2\pi}\left[\frac{\pi^3}{3} - \pi^3 + \pi^3\right]$$

$$= \frac{1}{2\pi} \times \frac{\pi^3}{3}$$

$$a_0 = \frac{\pi^2}{6}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi}\frac{1}{4}(x-\pi)^2\cos nx\,dx$$

$$= \frac{1}{2\pi}\int_0^{\pi}(x-\pi)^2\cos nx\,dx$$

$$= \frac{1}{2\pi}\left\{\frac{1}{n}\left[(x-\pi)^2\sin nx\right]_0^{\pi} - \frac{2}{n}\int_0^{\pi}(x-\pi)\sin nx\,dx\right]\right\}$$

$$= \frac{1}{2\pi}\left[0 - \frac{2}{n}\int_0^{\pi}(x-\pi)\sin nx\,dx\right]$$

$$= -\frac{1}{n\pi}\left\{(x-\pi)\sin nx\,dx\right.$$

$$= -\frac{1}{n\pi}\left\{(x-\pi)\int_0^{\pi}\sin nx\,dx - \int_0^{\pi}\left[1\cdot\int\sin nx\,dx\right]dx\right\}$$

$$= -\frac{1}{n\pi}\left\{-\frac{1}{n}\left[(x-\pi)\cos nx\right]_0^{\pi} + \left[\frac{1}{n^2}\sin nx\right]_0^{\pi}\right\}$$

$$= \frac{1}{n\pi}\left\{-\frac{1}{n}[0+\pi] + \frac{1}{n^2}(0)\right\}$$

$$= \frac{1}{n\pi}\left[-\frac{\pi}{n}\right]$$

$$a_n = \frac{1}{n^2}$$

$$① \Rightarrow \frac{1}{4}(x-\pi)^2 = \frac{\pi^2/6}{2} + \sum_{n=1}^{\infty} \frac{1}{n^2}\cos nx$$

$$\Rightarrow \frac{1}{4}(x-\pi)^2 = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{1}{n^2}\cos nx \quad — ②$$

when $x = 0$

$$② \Rightarrow \frac{\pi^2}{4} = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{1}{n^2} \quad — ①$$

$$② \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{4} - \frac{\pi^2}{12} = \frac{3\pi^2 - \pi^2}{12} = \frac{\pi^2}{6}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

$$\Rightarrow \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \frac{\pi^2}{6}$$

(ii) when $x = \pi$

$$② \Rightarrow 0 = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{1}{n^2} \cos n\pi$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = \frac{-\pi^2}{12}$$

$$\Rightarrow -\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} + \cdots \frac{-\pi^2}{12}$$

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} \cdots \frac{\pi^2}{12} \; //$$

③ Find the fourier series expansion of $f(x) = (\pi - x)^2$ in $[0, 2\pi]$.

Given $f(x) = (\pi - x)^2, \; x \in [0, 2\pi]$

$$f(x) = (x-\pi)^2$$

$$f(2\pi - x) = [2\pi - x - \pi]^2$$

$$= (\pi - x)^2$$

$$f(2\pi - x) = (x - \pi)^2$$

$f(x)$ is an even function.

$$\Rightarrow b_n = 0$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad — ①$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} (x-\pi)^2 dx$$

$$= \frac{2}{\pi}\int_0^{\pi} (x^2+\pi^2-2x\pi)dx$$

$$= \frac{2}{\pi}\left[\frac{x^3}{3}+\pi^2 x-\frac{2x^2}{2}\pi\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{\pi^3}{3}\right]$$

$$= \frac{2\pi^3}{3}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} (x-\pi)^2\cos nx\,dx$$

$$= \frac{2}{\pi}\left\{(x-\pi)^2\int_0^{\pi}\cos nx\,dx - \int_0^{\pi}\left[2(x-\pi)\int\cos nx\,dx\right]dx\right\}$$

$$= \frac{2}{\pi}\left\{\left[(x-\pi)^2\sin nx\right]_0^{\pi} - \frac{2}{n}(x-\pi)\sin nx\,dx\right\}$$

$$= \frac{2}{\pi}\left\{-0+\frac{2}{n^2}(x-\pi)\cos nx\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left\{-\frac{2}{n^2}(-\pi)\right\}$$

$$a_n = \frac{4}{n^2}$$

① $\Rightarrow$ $(x-\pi)^2 = \frac{2\pi^2}{6} + \sum_{n=1}^{\infty}\frac{4}{n^2}\cos nx$

$(x-\pi)^2 = \frac{2\pi^2}{6} + \sum_{n=1}^{\infty}\frac{4}{n^2}\cos nx$ — ②

when $x=0$

② $\Rightarrow$ $\pi^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2}$

$$\sum_{n=1}^{\infty}\frac{4}{n^2} = \pi^2 - \frac{\pi^2}{3}$$

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{2\pi^2}{12}$$

$$\Rightarrow \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\dots\dots\frac{\pi^2}{6}$$

when $x=\pi$

$$② \Rightarrow 0+\frac{\pi^2}{3}+\sum_{n=1}^{\infty}\frac{4}{n^2}\cos n\pi$$

$$\sum_{n=1}^{\infty}\frac{(-1)^n 4}{n^2} = -\frac{\pi^2}{3}$$

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} = \frac{\pi^2}{12}$$

$$\Rightarrow \frac{-1}{1^2}+\frac{1}{2^2}-\frac{1}{3^2}+\frac{1}{4^2}+\dots -\frac{\pi^2}{12}$$

$$\Rightarrow \frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}\dots\dots\frac{\pi^2}{12} //$$

④ Expand the function $f(x)=x(2\pi-x)$ in fourier series over the limits $[0,2\pi]$, hence deduce $\frac{\pi^2}{6}=\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\dots$

$\frac{\pi^2}{12}=\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\dots$

Given $f(x)=x(2\pi-x),\ x\in[0,2\pi]$

$$\Rightarrow f(2\pi-x) = (2\pi-x)(2\pi-2\pi+x)$$

$$= x(2\pi-x)$$

$f(x)$ is even function

$b_n=0$

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx \quad —①$$

$$a_0=\frac{2}{\pi}\int_0^{\pi}f(x)\,dx$$

$$=\frac{2}{\pi}\int_0^{\pi}x(2\pi-x)\,dx$$

$$=\frac{2}{\pi}\int_0^{\pi}[2\pi x-x^2]\,dx$$

$$=\frac{2}{\pi}\left[\pi x^2-\frac{x^3}{3}\right]_0^{\pi}$$

$$=\frac{2}{\pi}\left(\frac{2\pi^3}{3}\right)$$

$a_0 = \frac{4\pi^2}{3}$

$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx \, dx$

$= \frac{2}{\pi} \int_0^{\pi} (2\pi x - x^2) \cos nx \, dx$

$= \frac{2}{\pi} \left[(2\pi x - x^2) \int_0^{\pi} \cos nx \, dx - \int_0^{\pi} (2\pi - 2x) \int \cos nx dx \right] dx \}$

$= \frac{2}{\pi} \left[\frac{1}{n} (2\pi x - x^2) \sin nx \right]_0^{\pi} - \frac{2}{n^2} (\cos nx)_0^{\pi}$

$= \frac{2}{\pi} \left[\frac{1}{n} \int_0^{\pi} (2\pi - 2x) \sin nx \right] dx$

$= \frac{2}{\pi} \left[\frac{2}{n} \int_0^{\pi} (\pi - x) \sin nx \right] dx$

$= \frac{2}{\pi} \left[\frac{2}{n} (\pi - x) \cos nx \right]_0^{\pi}$

$= \frac{2}{\cancel{\pi}} \left[-\frac{2}{n^2} \cancel{\pi} \right]$

$a_n = \frac{-4}{n^2}$

① $\Rightarrow x(2\pi - x) = \frac{4\pi^2/3}{2} + \sum_{n=1}^{\infty} \left(\frac{-4}{n^2}\right) \cos nx$

$x(2\pi - x) = \frac{2\pi^2}{3} - \sum_{n=1}^{\infty} \frac{4}{n^2} \cos nx$ — ②

when $x = 0$

② $\Rightarrow \sum_{n=1}^{\infty} \frac{4}{n^2} \cos nx = \frac{2\pi^2}{3}$

$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{2\pi^2}{12}$

$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \frac{\pi^2}{6}$

when $x = \pi$

② $\Rightarrow \pi^2 = \frac{2\pi^2}{3} - \sum_{n=1}^{\infty} \frac{4}{n^2} \cos n\pi$

$\sum_{n=1}^{\infty} \frac{4}{n^2} \cos n\pi = \frac{2\pi^2}{3} - \pi^2$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}$$

$$-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \cdots \cdots -\frac{\pi^2}{12}$$

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} + \cdots \cdots \frac{\pi^2}{12} \;//$$

(5) obtain the fourier series of $f(x) = \begin{cases} x, & 0 \le x < \pi \\ 2\pi - x, & \pi \le x < 2\pi \end{cases}$ in $[0, 2\pi]$.

Given :- $f(x) = \begin{cases} x, & 0 \le x < \pi \\ 2\pi - x, & \pi \le x < 2\pi, \end{cases}$ $x \in [0, 2\pi]$

hence $\phi(x) = x$, $\psi(x) = 2\pi - x$

Let $\phi(2\pi - x) = 2\pi - x = \psi(x)$

$\therefore f(x)$ is an even function.

$\Rightarrow b_n = 0$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad -①$$

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x)\,dx$$

$$a_0 = \frac{2}{\pi} \int_0^{\pi} x\,dx$$

$$= \frac{2}{\pi} \left[\frac{x^2}{2}\right]_0^{\pi}$$

$$= \frac{1}{\pi} [\pi^2 - 0]$$

$$a_0 = \pi$$

$$\therefore a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx\,dx$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} x \cos nx\,dx$$

$$a_n = \frac{2}{\pi} \left\{ x \int_0^{\pi} \cos nx\,dx - \int_0^{\pi} \left[1 \cdot \int \cos nx\,dx\right] dx \right\}$$

$$a_n = \frac{2}{\pi}\left\{\frac{1}{n}(x\sin nx)_0^{\pi} + \frac{1}{n^2}(\cos nx)_0^{\pi}\right\}$$

$$a_n = \frac{2}{\pi}\left\{\frac{1}{n}(0-0) + \frac{1}{n^2}(\cos n\pi - 1)\right\}$$

$$a_n = \frac{2}{\pi n^2}\left[(-1)^n - 1\right]$$

$$a_n = \begin{cases} \frac{-4}{\pi n^2}, & \forall\ n = 1,3,5\ldots \\ 0, & \forall\ n = 2,4,6\ldots \end{cases}$$

$$(1) \Rightarrow f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2\left[(-1)^n - 1\right]\cos nx}{\pi n^2} \ //$$

(6) Obtain the fourier series of $f(x) = x^2$ in the interval $[-\pi, \pi]$ or find the fourier half range cosin series of $f(x) = x^2$ in the interval $[0, \pi]$.

Given $f(x) = x^2$, $x \in [-\pi, \pi]$

$$f(-x) = (-x)^2 = x^2 = f(x)$$

$f(x)$ is an even function

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad —(1)$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x^2\,dx$$

$$= \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi}$$

$$a_0 = \frac{2}{3}\pi^2$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx\,dx$$

$$= \frac{2}{\pi}\left\{x^2\int_0^{\pi}\cos nx\,dx - \int_0^{\pi}\left[2x\int \cos nx\,dx\right]dx\right\}$$

$$= \frac{2}{\pi}\left\{\frac{1}{\pi}[x^2 \sin nx]_0^\pi - \frac{2}{\pi}\int_0^\pi [x \sin nx]\,dx\right\}$$

$$= -\frac{4}{n\pi}\int_0^\pi x \sin nx\,dx$$

$$= -\frac{4}{n\pi}\left\{x\int_0^\pi \sin nx\,dx - \int_0^\pi [1 . \int \sin nx\,dx]\,dx\right\}$$

$$= -\frac{4}{n\pi}\left\{-\frac{1}{n}[x\cos nx]_0^\pi + \frac{1}{n^2}(\sin nx)_0^\pi\right\}$$

$$= -\frac{4}{n\pi}\left\{-\frac{1}{n}[\pi\cos n\pi - 0]\right\}$$

$$= \frac{4}{n^2\cancel{\pi}}[\cancel{\pi}(-1)^n]$$

$$a_n = \frac{4}{n^2}(-1)^n$$

(F) $\Rightarrow x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^n \cos nx$ //

(7) Obtain the fourier series $f(x) = |x|$ in the interval $[-\pi, \pi]$ and hence deduce the series $\frac{1}{1^2} + \frac{1}{3^2} + \dots + \frac{\pi^2}{8}$

Given $f(x) = |x| = \begin{cases} -x, & -\pi \le x \le 0 \\ x, & 0 \le x \le \pi \end{cases}$

$\phi(x) = -x \quad , \quad \psi(x) = x$

$\phi(-x) = -(-x) = x = \psi(x)$

$$f(x) = \frac{a_0}{2} + \sum_{n=0}^{\infty} a_n \cos nx \quad \text{—} (1)$$

$$a_0 = \frac{2}{\pi}\int_0^\pi f(x)\,dx$$

$$= \frac{2}{\pi}\int_0^\pi x\,dx$$

$$= \frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^\pi$$

$$= \frac{\cancel{2}}{\pi}\left[\frac{\pi^2}{\cancel{2}}\right]$$

$$a_0 = \pi$$

$$\therefore a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx$$

$$= \frac{2}{\pi}\left\{x\int_0^{\pi}\cos nx\,dx - \int_0^{\pi}\left[1\int\cos nx\,dx\right]dx\right\}$$

$$= \frac{2}{\pi}\left\{\frac{1}{n}\left[x\sin nx\right]_0^{\pi} + \frac{1}{n^2}(\cos nx)_0^{\pi}\right\}$$

$$= \frac{2}{\pi n^2}\left[\cos n\pi - 1\right]$$

$$= \frac{2}{\pi n^2}\left[(-1)^n - 1\right]$$

$$① \Rightarrow f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty}\frac{2}{\pi n^2}\left[(-1)^n - 1\right]\cos nx$$

$$\frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\left(\frac{(-1)^n - 1}{n^2}\right)\cos nx = \begin{cases} -x, & -\pi \le x \le 0 \\ x, & 0 \le x \le \pi \end{cases} \quad —②$$

when $x = 0$

$$② \Rightarrow \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n - 1}{n^2} = 0$$

$$\sum_{n=1}^{\infty}\frac{(-1)^n - 1}{n^2} = -\frac{\pi^2}{4}$$

$$-\frac{2}{1^2} + 0 - \frac{2}{3^2} + 0 - \frac{2}{5^2} + \cdots -\frac{\pi^2}{4}$$

$$-2\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right] = -\frac{\pi^2}{4}$$

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\cdots = \frac{\pi^2}{8} \,//$$

⑧ obtain the fourier series of $f(x) = \begin{cases} 1 + \frac{2x}{\pi}, & -\pi \le x \le 0 \\ 1 - \frac{2x}{\pi}, & 0 \le x \le \pi \end{cases}$ hence deduce the series $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\cdots$

Given $f(x) = \begin{cases} 1 + \frac{2x}{\pi}, & -\pi \le x \le 0 \\ 1 - \frac{2x}{\pi}, & 0 \le x \le \pi \end{cases}$

$\phi(x) = 1 + \frac{2x}{\pi}$, $\psi(x) = 1 - \frac{2x}{\pi}$

$\phi(x) = 1 - \frac{2x}{\pi} = \psi(x)$

$\therefore f(x)$ is an even function

$\Rightarrow b_n = 0$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \text{—} ①$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi}\left(1 - \frac{2x}{\pi}\right)dx$$

$$= \frac{2}{\pi}\left[x - \frac{x^2}{\pi}\right]_0^{\pi}$$

$$= \frac{2}{\pi}(0-0)$$

$$a_0 = 0$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi}\left(1 - \frac{2x}{\pi}\right)\cos nx\,dx$$

$$= \frac{2}{\pi}\left\{\left(1 - \frac{2x}{\pi}\right)\int_0^{\pi}\cos nx\,dx - \int_0^{\pi}\left[\left(\frac{-2}{\pi}\right)\int \cos nx\,dx\right]dx\right\}$$

$$= \frac{2}{\pi}\left\{\frac{1}{n}\left(1 - \frac{2x}{\pi}\right)\sin nx - \int_0^{\pi}\frac{2}{n^2\pi}\left(\cos nx\right)_0^{\pi}\right\}$$

$$= \frac{2}{n}\left\{\frac{1}{n}(0-0) - \frac{2}{n^2\pi}(\cos n\pi - 1)\right\}$$

$$= \frac{4}{n^2\pi^2}\left\{(-1)^n - 1\right\}$$

$$a_n = \frac{4}{n^2\pi^2}\left[1 - (-1)^n\right]$$

$$f(x) = \frac{0}{2} + \sum_{n=1}^{\infty}\frac{4}{n^2\pi^2}\left[1 - (-1)^n\right]\cos nx$$

$$\frac{4}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{1-(-1)^n}{n^2}\right]\cos nx = \begin{cases} 1+\frac{2x}{\pi}, & -\pi \le x \le 0 \\ 1-\frac{2x}{\pi}, & 0 \le x \le \pi \end{cases} \quad -② $$

when $x = 0$

$$② \Rightarrow \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n^2} = 1$$

$$\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n^2} = \frac{\pi^2}{4}$$

$$\frac{2}{1^2} + 0 + \frac{2}{3^2} + 0 + \frac{2}{5^2} + 0 + \dots \frac{\pi^2}{4}$$

$$2\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2}\right] + \dots \frac{\pi^2}{4}$$

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots \frac{\pi^2}{8} \;//$$

9. Obtain the fourier series for the function $f(x) = \begin{cases} -\pi, & -\pi \le x < 0 \\ x, & 0 \le x < \pi \end{cases}$

Given $f(x) = \begin{cases} -\pi, & -\pi \le x < 0 \\ x, & 0 \le x < \pi \end{cases}$

$\phi(x) = -\pi \qquad \psi(x) = x$

$\phi(-x) \neq \psi(x)$ & $\phi(-x) \neq -\psi(x)$

The given function is neither even nor odd

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n\cos nx + b_n\sin nx] \quad -①$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\,dx + \int_{0}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0}(-\pi)\,dx + \int_{0}^{\pi} x\,dx$$

$$= \frac{1}{\pi}\left\{-\pi[x]_{-\pi}^{0} + \left(\frac{x^2}{2}\right)_0^{\pi}\right\}$$

$$= \frac{1}{\pi}\left\{-\pi(0+\pi)+\frac{\pi^2}{2}-0\right\}$$

$$= \frac{1}{\pi}\left[-\pi^2+\frac{\pi^2}{2}\right]$$

$$= \frac{1}{\pi}\left[-\frac{\pi^2}{2}\right]$$

$$a_0 = -\frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\cos nx\,dx + \int_{0}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[-\pi\int_{-\pi}^{0}\cos nx\,dx + \int_{0}^{\pi} x\cos nx\,dx\right.$$

$$= \frac{1}{\pi}\left\{-\frac{\pi}{n}(\sin nx]_{-\pi}^{0} + \int_{0}^{\pi} x\cos nx\,dx\right\}$$

$$= \frac{1}{\pi}\int_{0}^{\pi} x\cos nx\,dx$$

$$a_n = \frac{1}{\pi}\left\{\frac{1}{n}(x\sin x)\Big|_0^{\pi} + \frac{1}{n^2}\cos nx\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left\{\frac{1}{n}(0-0)+\frac{1}{n^2}[\cos n\pi-1]\right\}$$

$$= \frac{1}{n^2\pi}\left[(-1)^n-1\right]$$

$$a_n = \frac{(-1)^n-1}{n^2\pi}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\sin nx\,dx + \int_{0}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left\{-\pi\int_{-\pi}^{0}\sin nx\,dx + \int_{0}^{\pi} x\sin nx\,dx\right.$$

$$= \frac{1}{\pi}\left\{\frac{\pi}{n}[\cos nx]_{-\pi}^{0} - \frac{1}{n}(x\cos nx]_0^{\pi} + \frac{1}{n^2}(\sin nx)_0^{\pi}\right.$$

$$= \frac{1}{\pi}\left\{\frac{\pi}{n}(1-\cos n\pi) - \frac{1}{n}[\pi\cos n\pi - 0] + \frac{1}{n^2}(0-0)\right\}$$

$$= \frac{1}{\pi}\left\{ \frac{\pi}{n} - \frac{\pi}{n}\cos n\pi - \frac{\pi}{n}\cos n\pi \right]$$

$$= \frac{1}{\pi}\left\{ \frac{\pi}{n} - \frac{2\pi}{n}\cos n\pi \right]$$

$$= \frac{1}{n} - \frac{2}{n}\cos n\pi$$

$$b_n = \frac{1-2(-1)^n}{n}$$

$$f(x) = -\frac{\pi}{4} + \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2\pi}\cos nx + \left[\frac{1-2(-1)^n}{n}\right]\sin nx. //$$

(10) Find the fourier Series of $f(x) = \begin{cases} 0, & -\pi \le x < 0 \\ x, & 0 \le x < \pi \end{cases}$

Given $f(x) = \begin{cases} 0, & -\pi \le x < 0 \\ x, & 0 \le x < \pi \end{cases}$

$\therefore f(x)$ is either even nor odd

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos nx + b_n \sin nx\right] \quad \text{—①}$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\left\{\int_{-\pi}^{0} f(x)\,dx + \int_{0}^{\pi} f(x)\,dx\right\}$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\,dx + \int_{0}^{\pi} x\,dx\right]$$

$$= \frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{\pi^2}{2}\right]$$

$$a_0 = \frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\cos nx\,dx + \int_{0}^{\pi} f(x)\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\int_{0}^{\pi} x\cos nx\,dx$$

$$= \frac{1}{\pi}\left[x\int_0^{\pi}\cos nx\,dx - \int_0^{\pi}\left[1\cdot\int\cos nx\,dx\right]dx\right\}$$

$$= \frac{1}{\pi}\left\{\frac{1}{n}(x\sin nx)_0^{\pi} + \frac{1}{n^2}[\cos nx]_0^{\pi}\right\}$$

$$= \frac{1}{\pi}\left\{\frac{1}{n}(0-0) + \frac{1}{n^2}(\cos n\pi - 1)\right\}$$

$$a_n = \frac{1}{\pi}\left\{\frac{1}{n^2}(-1)^n - 1\right\}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\sin nx\,dx + \int_0^{\pi} f(x)\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\int_0^{\pi} x\sin nx\,dx$$

$$= \frac{1}{\pi}\left\{x\int_0^{\pi}\sin nx\,dx - \int_0^{\pi}\left[1\int\sin nx\,dx\right]dx\right\}$$

$$= \frac{1}{\pi}\left\{-\frac{1}{n}(x\cos nx)_0^{\pi} + \frac{1}{n^2}(\sin nx)_0^{\pi}\right\}$$

$$= \frac{1}{\pi}\left\{\frac{-1}{n}[\pi\cos n\pi - 0] + \frac{1}{n^2}(0-0)\right\}$$

$$b_n = \frac{-\pi\cos n\pi}{n\pi}$$

$$b_n = \frac{-(-1)^n}{n}$$

$$b_n = \frac{(-1)^{n+1}}{n}$$

$$① \Rightarrow f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty}\left[\frac{1}{n^2}(-1)^n - 1\cos nx + \frac{(-1)^{n+1}}{n}\sin nx\right]$$

11. Find the fourier series $f(x) = x - x^2,\ x \in [-\pi, \pi]$.

Given $f(x) = x - x^2,\ x \in [-\pi, \pi]$

$\therefore f(x)$ is neither even nor odd

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n\cos nx + b_1\sin nx] \quad - ①$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} (x - x^2)\, dx$$

$$= \frac{1}{\pi}\left(\frac{x^2}{2} - \frac{x^3}{3}\right)_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left\{\left[\frac{\pi^2}{2} - \frac{\pi^3}{3}\right] - \left[\frac{\pi^2}{2} + \frac{\pi^3}{3}\right]\right\}$$

$$= \frac{1}{\pi}\left[-\frac{2\pi^3}{3}\right]$$

$$a_0 = \frac{-2}{3}\pi^2$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} (x - x^2)\cos nx\, dx$$

$$= \frac{1}{\pi}\left\{(x - x^2)\int_{-\pi}^{\pi}\cos nx - \int_{-\pi}^{\pi}(1-2x)\int \cos nx\, dx\right\} dx$$

$$= \frac{1}{\pi}\left[(1-2x)\int_{-\pi}^{\pi}\sin nx\, dx\, t\, (1-2x)\right]$$

$$= -\frac{1}{n\pi}\int_{-\pi}^{\pi}(1-2x)\sin nx\, dx$$

$$= \frac{-1}{n\pi}\left\{-\frac{1}{\pi}(1-2x)\cos nx \int_{-\pi}^{\pi} \frac{-2}{n^2}(\sin nx)_{-\pi}^{\pi}\right\}$$

$$= \frac{-1}{n\pi}\left\{-\frac{1}{n}(1-2\pi)\cos n\pi) - (1+2\pi)\cos n\pi] - 0\right\}$$

$$= \frac{-1}{n\pi}\left\{-\frac{1}{n}(\cancel{\cos} n\pi - 2\pi\cos n\pi - \cos n\pi - 2\pi\cos n\pi)\right.$$

$$= \frac{-1}{n\pi}\left\{\frac{-1}{n}(-4\pi\cos n\pi)\right\}$$

$$= -\frac{4\cancel{\pi}\cos n\pi}{n^2\cancel{\pi}}$$

$$= \frac{-4}{n^2}\cos n\pi$$

$$= \frac{-4}{n^2}(-1)^n$$

$$= \frac{4}{n^2}(-1)^{n+1}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} (x-x^2)\sin nx\,dx$$

$$= \frac{1}{\pi}\left\{(x-x^2)\int_{-\pi}^{\pi}\sin nx\,dx - \int_{-\pi}^{\pi}(1-2x)\left[\int \sin nx\,dx\right]dx\right\}$$

$$= \frac{1}{\pi}\left\{\frac{-1}{n}(x-x^2)\cos n\pi\Big]_{-\pi}^{\pi} + \frac{1}{n}\int_{-\pi}^{\pi}(1-2x)\cos nx\,dx\right\}$$

$$= \frac{1}{\pi}\left\{-\frac{1}{n}\left[(\pi-\pi^2)\cos n\pi - (-\pi-\pi^2)\cos n\pi\right] + \frac{1}{n}\int_{-\pi}^{\pi}(1-2x)\cos nx\,dx\right\}$$

$$= \frac{1}{\pi}\left\{-\frac{1}{n}\left[2\pi\cos n\pi\right] + \frac{1}{n}I_1\right\}$$

$$b_n = \frac{1}{\pi}\left[\frac{-2\pi\cos n\pi}{n} + \frac{1}{n}I_1\right]$$

$$I_1 = \int_{-\pi}^{\pi}(1-2x)\cos nx\,dx$$

$$= \frac{1}{n}\left[(1-2x)\sin nx \int_{-\pi}^{\pi} -\frac{2}{n^2}\cos nx\right]_{-\pi}^{\pi}$$

$$\Rightarrow I_1 = 0$$

$$b_n = -\frac{2\pi\cos n\pi}{n\pi}$$

$$= -\frac{2}{n}\cos n\pi$$

$$= -\frac{2}{n}(-1)^n$$

$$b_n = \frac{2}{n}(-1)^{n+1}$$

$$① \Rightarrow (x-x^2) = \frac{-\pi^2}{3} + \sum_{n=1}^{\infty}\left[\frac{4}{n^2}(-1)^{n+1}\cos nx + \frac{2}{n}(-1)^{n+1}\sin nx\right] \quad ②$$

when $x=0$

$$② \Rightarrow 0 = -\frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^{n+1}$$

$$4\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{3}$$

$$4\left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots\right] = \frac{\pi^2}{3}$$

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12} \;//$$

⑫ Find the fourier series $f(x)=x^2\,[0,\pi]$ or $f(x)=x^2\,[-\pi,\pi]$

Given $f(x)=x^2,\ x\in[0,\pi]$

WKT

The fourier half-range cosine series is

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx \quad \text{—①}$$

$$a_0=\frac{2}{\pi}\int_0^{\pi}f(x)\,dx$$

$$=\frac{2}{\pi}\int_0^{\pi}x^2\,dx$$

$$=\frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi}$$

$$a_0=\frac{2}{3}\pi^2$$

$$a_n=\frac{2}{\pi}\int_0^{\pi}f(x)\cos nx\,dx$$

$$=\frac{2}{\pi}\int_0^{\pi}x^2\cos nx\,dx$$

$$=\frac{2}{\pi}\left\{x^2\int_0^{\pi}\cos nx\,dx-\int_0^{\pi}\left[2x\int\cos nx\,dx\right]dx\right\}$$

$$=\frac{2}{\pi}\left\{\frac{1}{\pi}\left[x^2\sin nx\right]_0^{\pi}-\frac{2}{\pi}\int_0^{\pi}(x\sin x)\,dx\right\}$$

$$=-\frac{4}{n\pi}\int_0^{\pi}x\sin nx\,dx$$

$$=-\frac{4}{n\pi}\left\{x\int_0^{\pi}\sin nx\,dx-\int_0^{\pi}\left[1\int\sin nx\,dx\right]dx\right\}$$

$$=-\frac{4}{n\pi}\left\{-\frac{1}{n}\left[x\cos n\pi\right]_0^{\pi}+\frac{1}{n^2}\left[\sin nx\right]_0^{\pi}\right\}$$

$$=-\frac{4}{n\pi}\left\{-\frac{1}{n}\left[\pi\cos n\pi-0\right]\right\}$$

$$=\frac{4}{n^2\pi}\left[\pi(-1)^n\right]$$

$$a_n=\frac{4}{n^2}(-1)^n$$

$$① \Rightarrow x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^n \cos nx$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}$$

$$-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} + \cdots \quad \frac{\pi^2}{12}$$

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots \quad \frac{\pi^2}{12}$$

(13) Find the fourier half range cosine & sin series of $f(x) = x[\pi - x]$ in the $[0, \pi]$

Given $f(x) = x(\pi - x) = \pi x - x^2,\ x \in [0, \pi]$

WKT

The fourier half range cosine series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad — ①$$

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x)\,dx$$

$$= \frac{2}{\pi} \int_0^{\pi} (\pi x - x^2)\,dx$$

$$= \frac{2}{\pi} \left[\frac{\pi x^2}{2} - \frac{x^3}{3}\right]_0^{\pi}$$

$$= \frac{2}{\pi} \left[\frac{\pi^3}{6}\right]$$

$$= \frac{2\pi^2}{6}$$

$$a_0 = \frac{\pi^2}{3}$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx\,dx$$

$$= \frac{2}{\pi} \int_0^{\pi} (\pi x - x^2) \cos nx\,dx$$

$$= \frac{2}{\pi} \left\{ (\pi x - x^2) \int_0^{\pi} \cos nx\,dx - \int_0^{\pi} (\pi - 2x) \left[\int \cos nx\,dx\right] dx \right\}$$

$$= \frac{2}{\pi} \left\{ \frac{1}{\pi} (\pi x - x^2) \sin nx \Big]_0^{\pi} - \frac{1}{n} \int_0^{\pi} (\pi - 2x) \sin nx\,dx \right\}$$

$$= -\frac{2}{n\pi}\int_0^{\pi}(\pi-2x)\sin nx\,dx$$

$$= \frac{-2}{n\pi}\left\{(\pi-2x)\int_0^{\pi}\sin nx\,dx - \int_0^{\pi}(-2)\left[\int \sin nx\,dx\right]dx\right\}$$

$$= \frac{-2}{n\pi}\left\{-\frac{1}{n}\left[(\pi-2x)\cos nx\right]_0^{\pi} - \frac{2}{n^2}\left[\sin nx\right]_0^{\pi}\right\}$$

$$= \frac{-2}{n\pi}\left\{\frac{-1}{n}\left[-\pi\cos n\pi-\pi\right] - \frac{2}{n^2}[0-0]\right\}$$

$$= \frac{-2}{n\pi}\left\{\frac{\pi}{n}(1+\cos n\pi)\right\}$$

$$a_n = -\frac{2}{n\pi}\left[1+(-1)^n\right]$$

$$① \Rightarrow x(\pi-x) = \frac{\pi^2}{6} - 2\sum_{n=1}^{\infty}\left[\frac{1+(-1)^n}{n^2}\right]\cos nx$$

WKT

The fourier half range sin series is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \quad — ②$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi}(\pi x-x^2)\sin nx\,dx$$

$$= \frac{2}{\pi}\left\{(\pi x-x^2)\int_0^{\pi}\sin nx\,dx - \int_0^{\pi}\left[(\pi-2x)\int \sin nx\,dx\right]dx\right\}$$

$$= \frac{2}{\pi}\left\{\frac{-1}{n}\left[(\pi x-x^2)\cos nx\right]_0^{\pi} + \frac{1}{n}\int_0^{\pi}(\pi-2x)\cos nx\,dx\right\}$$

$$= \frac{2}{\pi}\int_0^{\pi}(\pi-2x)\cos nx\,dx$$

$$= \frac{2}{n\pi}\left\{(\pi-2x)\int_0^{\pi}\cos nx\,dx - \int_0^{\pi}(-2)\left[\int \cos nx\,dx\right]dx\right\}$$

$$= \frac{2}{n\pi}\left\{\frac{1}{n}\left[(\pi-2x)\sin nx\,dx\right]_0^{\pi} - \frac{2}{n^2}\left[\cos nx\right]_0^{\pi}\right\}$$

$$= \frac{2}{n\pi}\left\{\frac{1}{n}(0-0) - \frac{2}{n^2}\left[\cos n\pi - \cos 0\right]\right\}$$

$$= \frac{-4}{n^3\pi}\left[(-1)^n-1\right]$$

$$b_n = \frac{4}{n^3\pi}\left[1-(-1)^n\right]\sin nx \,//$$

(14) Expand the function $f(x) = x\sin x$ as a fourier series in the interval $0 \le x \le \pi$, deduce that $\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \frac{1}{7.9} + \dots = \frac{\pi - 2}{4}$

$f(x) = x\sin x,\ x \in [0, \pi]$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \text{—} ①$$

$$= \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x \sin x \cos nx\, dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x\,[2\cos nx \cdot \sin x]\, dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x\,[\sin(n+1)x - \sin(n-1)x\, dx]$$

$$= \frac{1}{\pi}\int_0^{\pi} x \sin(n+1)x\, dx - \frac{1}{\pi}\int_0^{\pi} x\sin(n-1)x\, dx$$

$$\int_0^{\pi} x\sin(n+1)x\, dx = -\frac{1}{n+1}\left[x\cos(n+1)x\right]_0^{\pi} + \frac{1}{(n+1)^2}\left[\sin(n+1)x\right]_0^{\pi}$$

$$= \frac{-1}{n+1}\left[\pi\cos(n+1)\pi - 0\right]$$

$$= \frac{-\pi}{n+1}\left[\cos(n+1)\pi\right]$$

$$= \frac{-\pi(-1)^{n+1}}{n+1}$$

$$= \frac{\pi(-1)^n}{n+1}$$

$$a_n = \frac{1}{\cancel{\pi}}\,\frac{\cancel{\pi}(-1)^n}{n+1} - \frac{1}{\cancel{\pi}}\,\frac{\cancel{\pi}(-1)^n}{n-1}$$

$$= (-1)^n\left[\frac{1}{n+1} - \frac{1}{n-1}\right]$$

$$= (-1)^n\left[\frac{n-1-n+1}{n^2+1}\right]$$

$$a_n = \frac{-2(-1)^n}{n^2-1} \quad for \quad n \geq 2$$

$$a_1 = \frac{2}{\pi}\int_0^\pi f(x)\cos x\,dx$$

$$= \frac{2}{\pi}\int_0^\pi x\sin x\cos x\,dx$$

$$= \frac{1}{\pi}\int_0^\pi x\,[2\sin x\cos x]\,dx$$

$$= \frac{1}{\pi}\int_0^\pi x\sin 2x\,dx$$

$$= \frac{1}{\pi}\left\{-\frac{1}{2}[x\cos 2x]_0^\pi + \frac{1}{4}[\sin 2x]_0^\pi\right\}$$

$$= \frac{1}{\pi}\left\{\frac{-1}{2}[\pi - 0]\right.$$

$$= \frac{1}{\pi}\left[\frac{-\pi}{2}\right]$$

$$= \frac{-1}{2}$$

$$① \Rightarrow f(x) = \frac{a_0}{2} + a_1\cos x + \sum_{n=2}^{\infty} a_n\cos nx$$

$$x\sin x = 1 - \frac{1}{2}\cos x - 2\sum_{n=2}^{\infty}\frac{(-1)^n}{n^2-1}\cos nx \quad - ②$$

Let $x = \frac{\pi}{2}$

$$② \Rightarrow \frac{\pi}{2}\sin\left(\frac{\pi}{2}\right) = 1 - \frac{1}{2}\cos\left(\frac{\pi}{2}\right) - 2\sum_{n=2}^{\infty}\frac{(-1)^n}{n^2-1}\cos\left(\frac{n\pi}{2}\right)$$

$$\frac{\pi}{2} = 1 - 2\left[\frac{1}{3}\cos\pi - \frac{1}{8}\cos\frac{3\pi}{2} + \frac{1}{15}\cos(2\pi) - \frac{1}{24}\cos\left(\frac{5\pi}{2}\right)\right.$$

$$+ \frac{1}{35}\cos(3\pi) + \ldots\ldots$$

$$= -2\left[\frac{-1}{3} - 0 + \frac{1}{15} - 0 - \frac{1}{35} + \ldots\ldots\right] = \frac{\pi}{2} - 1$$

$$= -2\left[\frac{-1}{3} + \frac{1}{15} - \frac{1}{35} + \ldots\right] = \frac{\pi - 2}{2}$$

$$= \frac{1}{3} - \frac{1}{15} + \frac{1}{35} \ldots\ldots\ldots = \frac{\pi - 2}{4}$$

$$= \frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \ldots\ldots = \frac{\pi - 2}{4} \;\|$$

(15) Find the fourier series of $f(x) = \sqrt{1-\cos x}$ in the interval $[-\pi, \pi]$ or $[0, 2\pi]$ and hence deduce $\sum_{n=1}^{\infty} \frac{1}{4n^2-1} = \frac{1}{2}$

Let $f(x) = \sqrt{1-\cos x} = \sqrt{2\sin^2\left(\frac{x}{2}\right)} = \sqrt{2}\,\sin\left(\frac{x}{2}\right), x \in [-\pi, \pi]$

$$f(-x) = \sqrt{1-\cos(-x)}$$
$$= \sqrt{1-\cos x}$$
$$= f(x)$$

$f(x)$ is an even function

$b_n = 0$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \text{—} ①$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$
$$= \frac{2}{\pi}\int_0^{\pi} \sqrt{2}\,\sin\left(\frac{x}{2}\right)dx$$
$$= \frac{2\sqrt{2}}{\pi}\left[\frac{-\cos(x/2)}{1/2}\right]_0^{\pi}$$

$f(-x) = \sqrt{1-\cos(-x)} = \sqrt{1-\cos x} = f(x)$

$f(x)$ is an even function.

$b_n = 0$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \text{—} ①$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$
$$= \frac{2}{\pi}\int_0^{\pi} \sqrt{2}\,\sin\left(\frac{x}{2}\right)dx$$
$$= \frac{2\sqrt{2}}{\pi}\left[\frac{-\cos(x/2)}{1/2}\right]_0^{\pi}$$
$$= -\frac{4\sqrt{2}}{\pi}\left[\cos\left(\frac{x}{2}\right)\right]_0^{\pi}$$
$$= -\frac{4\sqrt{2}}{\pi}\left[\cos\frac{\pi}{2} - \cos 0\right]$$

$$= -\frac{4\sqrt{2}}{\pi}\,[0-1]$$

$$a_0 = \frac{4\sqrt{2}}{\pi}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi}\sqrt{2}\,\sin\left(\frac{x}{2}\right)\cos nx\,dx$$

$$= \frac{\sqrt{2}}{\pi}\int_0^{\pi} 2\cos nx\,\sin\left(\frac{x}{2}\right)dx$$

$$= \frac{\sqrt{2}}{\pi}\int_0^{\pi}\left[\sin\left(nx+\frac{x}{2}\right) - \sin\left(nx-\frac{x}{2}\right)\right]dx$$

$$= \frac{\sqrt{2}}{\pi}\int_0^{\pi}\left[\sin\left(n+\frac{1}{2}\right)x - \sin\left(n-\frac{1}{2}\right)\right]dx$$

$$= \frac{\sqrt{2}}{\pi}\left[\frac{-\cos(n+1/2)x}{n+1/2} + \frac{\cos(n-1/2)x}{n-1/2}\right]_0^{\pi}$$

$$= \frac{\sqrt{2}}{\pi}\left\{(0-0) - \left[-\frac{1}{n+1/2} + \frac{1}{n-1/2}\right]\right\}$$

$$= -\frac{\sqrt{2}}{\pi}\left[\frac{1}{n-1/2} - \frac{1}{n+1/2}\right]$$

$$= \frac{-\sqrt{2}}{\pi}\left[\frac{1}{\frac{2n-1}{2}} - \frac{1}{\frac{2n+1}{2}}\right]$$

$$= \frac{-2\sqrt{2}}{\pi}\left[\frac{1}{2n-1} - \frac{1}{2n+1}\right]$$

$$= -2\sqrt{2}\left[\frac{2n+1-2n+1}{4n^2-1}\right]$$

$$a_n = \frac{-4\sqrt{2}}{\pi}\left[\frac{1}{4n^2-1}\right]$$

① $\Rightarrow$ $$\sqrt{1-\cos x} = \frac{2\sqrt{2}}{\pi} - \frac{4\sqrt{2}}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}\cos nx$$

$x = 0$

② $$0 = \frac{2\sqrt{2}}{\pi} - \frac{4\sqrt{2}}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}$$

$$\frac{4\sqrt{2}}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1} = \frac{2\sqrt{2}}{\pi}$$

$$\sum_{n=1}^{\infty}\frac{1}{4n^2-1} = \frac{2\sqrt{2}}{\pi}\cdot\frac{\pi}{4\sqrt{2}} = \frac{1}{2}\ //$$

Fourier Series having the period $2l$:-

The Fourier Series of $f(x) = [c, c+2l]$, over the period $2l$ is defined as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right)\right]$$

$$a_0 = \frac{1}{l}\int_{c}^{c+2l} f(x)\,dx$$

$$a_n = \frac{1}{l}\int_{c}^{c+2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$b_n = \frac{1}{l}\int_{c}^{c+2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

Note:-

① If $c=0$, we can define $f(x)$ in the interval $[0, 2l]$ for

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right)\right]$$

$$a_0 = \frac{1}{l}\int_{0}^{2l} f(x)\,dx$$

$$a_n = \frac{1}{l}\int_{0}^{2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$b_n = \frac{1}{l}\int_{0}^{2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

② If $c=-\pi$, we can define $f(x)$ in the interval $[-\pi, \pi]$ for

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

③ If $f(x)$ is an even function in the intervals $[0, 2l]$ or $[-l, l]$, then $b_n = 0$, we have

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$ and it also called the

Fourier half-range cosine series in $[0, l]$

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

(4) If $f(x)$ is an odd function in $[0, 2l]$ or $[-l, l]$, then $a_0 = 0$, $a_n = 0$, we have

$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$ and it is also called the Fourier half range sin series in $[0, l]$.

When $b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$

(5) If $f(x)$ is continuous in $[0, 2l]$, then $f(x)$ is even when $f(2l - x) = f(x)$, $f(x)$ is odd then, $f[2l - x] = -f(x)$

(6) $F(x) = \begin{cases} \phi(x), & 0 \le x \le l \\ \psi(x), & l \le x \le 2l \end{cases}$, then

$\phi(2l - x) = \psi(x)$ is even function

$\phi(2l - x) = -\psi(x)$ is odd function

(7) If $f(x)$ is continuous $[-l, l]$, then $f(x)$ is even $f(-x) = f(x)$ and odd when $f(-x) = -f(x)$.

(8) $f(x) = \begin{cases} \phi(x), & -l \le x \le 0 \\ \psi(x), & 0 \le x \le l \end{cases}$ then

$\phi(-x) = \psi(x)$ is even function

$\phi(-x) = -\psi(x)$ is odd function.

① Find the fourier series of $f(x) = x(2-x)$ in the interval $(0, 2)$, hence deduce $\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$

Given $f(x) = x(2-x)$, $x \in [0, 2]$

$l = 1$

$\Rightarrow f(2-x) = (2-x)[2-(2-x)]$

$\Rightarrow f(2-x) = (2-x)[2-2+x]$

$\Rightarrow f(2-x) = x(2-x) = f(x)$

$f(x)$ is an even function

$\Rightarrow b_n = 0$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) \quad -①$$

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx$$

$$= \frac{2}{l}\int_0^l x(2-x)\,dx$$

$$= 2\left[x^2 - \frac{x^3}{3}\right]_0^{l=1}$$

$$= 2\left[1 - \frac{1}{3}\right]$$

$$= 2\left[\frac{2}{3}\right]$$

$$a_0 = \frac{4}{3}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= 2\int_0^1 f(x)\cos(n\pi x)\,dx$$

$$= 2\int_0^1 (2x - x^2)\cos(n\pi x)\,dx$$

$$= 2\left\{(2x-x^2)\int_0^1 \cos(n\pi x)\,dx - \int_0^1 \left[(2-2x)\int \cos(n\pi x)\,dx\right]dx\right\}$$

$$= 2\left\{\frac{1}{n\pi}\left[(2x-x^2)\sin(n\pi x)\right]_0^1 - \frac{2}{n\pi}\int_0^1 (1-x)\sin(n\pi x)\,dx\right\}$$

$$= \frac{-4}{n\pi} \int_0^1 (1-x)\sin(n\pi x)\,dx$$

$$= \frac{4}{n\pi} \int_0^1 (x-1)\sin(n\pi x)\,dx$$

$$= \frac{4}{n\pi} \left\{ (x-1) \int_0^1 \sin(n\pi x)\,dx - \int_0^1 \left[1 \int \sin(n\pi x)\,dx \right] dx \right.$$

$$= \frac{4}{n\pi} \left\{ \frac{-1}{n\pi} \left[(x-1)\cos(n\pi x) \right]_0^1 + \frac{1}{n^2\pi^2} \left[\sin(n\pi x) \right] dx \right.$$

$$= \frac{-4}{n^2\pi^2} [0+1]$$

$$a_n = -\frac{4}{n^2\pi^2}$$

$$\therefore ① \Rightarrow x(2-x) = \frac{2}{3} - \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \cos(n\pi x) \quad — ②$$

$$0 = \frac{2}{3} - \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\Rightarrow \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{2}{3}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{2}{3} \times \frac{\pi^2}{4}$$

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}$$

② Find the fourier series of $f(x) = \begin{cases} \pi x, & 0 \le x < 1 \\ \pi(2-x), & 1 \le x < 2 \end{cases}$

Given $f(x) = \begin{cases} \pi x, & 0 \le x < 1 \\ \pi(2-x), & 1 \le x < 2 \end{cases}$ $\quad 2l = 2$

$l = 1$

$$f(x) = \begin{cases} \pi x, & 0 \le x < l \\ \pi(2-x), & l \le x < 2l \end{cases}$$

$$\phi(x) = \pi x, \quad \psi(x) = \pi(2-x)$$

$$\phi(2-x) = \pi(2-x) = \psi(x)$$

$f(x)$ is an even function, $\Rightarrow b_n = 0$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) \quad — ①$$

$$\therefore a_0 = \frac{2}{l}\int_0^l f(x)\,dx$$

$$= \frac{2}{1}\int_0^1 \pi x\,dx$$

$$= 2\pi\int_0^1 x\,dx$$

$$= 2\pi\left[\frac{x^2}{2}\right]_0^1$$

$$= \cancel{2}\pi\left[\frac{1}{\cancel{2}} - 0\right]$$

$$a_0 = \pi$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{1}\int_0^1 f(x)\cos(n\pi x)\,dx$$

$$= 2\int_0^1 \pi\, x\cos(n\pi x)\,dx$$

$$= 2\pi\int_0^1 x\cos(n\pi x)\,dx$$

$$= 2\pi\left\{x\int_0^1 \cos(n\pi x)\,dx - \int_0^1 \left[1\int\cos(n\pi x)\,dx\right]dx\right\}$$

$$= 2\pi\left\{\frac{1}{2\pi}\left[x\sin(n\pi x)\right]_0^1 + \frac{1}{n^2\pi^2}\left[\cos(n\pi x)\right]_0^1\right\}$$

$$= \frac{2\pi}{n^2\pi^2}\left[\cos n\pi - 1\right]$$

$$a_n = \frac{2}{n^2\pi}\left[(-1)^n - 1\right]$$

$$① = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{n^2}\right]\cos(n\pi x) = \begin{cases}\pi x, & 0 \le x < 1\\ \pi(2-x), & 1 \le x < 2\end{cases}$$

$$\sum_{n=1}^{\infty}\frac{(-1)^n - 1}{n^2} = -\frac{\pi^2}{4}$$

$$\Rightarrow -\frac{\pi^2}{4} = \frac{(-2)}{1^2} + \frac{(-2)}{3^2} + \frac{(-2)}{5^2} + \cdots$$

$$-\frac{\pi^2}{4} = (-2)\left(\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right)$$

$$\frac{\pi^2}{8} = \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots \quad //$$

③ Find the fourier series of $f(x) = \begin{cases} l+x, & -l \le x \le 0 \\ l-x, & 0 \le x \le l \end{cases}$

Given $f(x) = \begin{cases} l+x, & -l \le x \le 0 \\ l-x, & 0 \le x \le l \end{cases}$

$\phi(x) = l + x \quad , \quad \psi(x) = l - x$

$\therefore \phi(-x) = l - x = \psi(x)$

$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) \quad -①$

$a_0 = \frac{2}{l}\int_0^l f(x)\,dx$

$= \frac{2}{l}\int_0^l (l-x)\,dx$

$= \frac{2}{l}\left[l^2 - \frac{l^2}{2}\right]$

$= \frac{2}{l}\left[\frac{l^2}{2}\right]$

$a_0 = l$

$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$

$= \frac{2}{l}\int_0^l (l-x)\cos\left(\frac{n\pi x}{l}\right)dx$

$= \frac{2}{l}\left\{(l-x)\int_0^l \cos\left(\frac{n\pi x}{l}\right)dx - \int_0^l (-1)\left[\int \cos\left(\frac{n\pi x}{l}\right)dx\right]dx\right\}$

$= \frac{2}{l}\left\{\frac{l}{n\pi}(l-x)\sin\left(\frac{n\pi x}{l}\right)\Big]_0^l - \frac{l^2}{n^2\pi^2}\left[\cos\left(\frac{n\pi x}{l}\right)\right]_0^l\right\}$

$= \frac{2}{l}\left\{0 - \frac{l^2}{n^2\pi^2}(\cos n\pi - 1)\right\}$

$= \frac{2l^2}{l n^2\pi^2}((-1)^n - 1)$

$= \frac{-2l}{n^2\pi^2}((-1)^n - 1)$

$a_n = \frac{2l}{n^2\pi^2}(1 - (-1)^n)$

$$① \Rightarrow \frac{l}{2} + \frac{2l}{\pi^2} \sum_{n=1}^{\infty} \left(\frac{1-(-1)^n}{n^2}\right) \cos\left(\frac{n\pi x}{l}\right) = \begin{cases} l+x, & -l \le x \le 0 \\ l-x, & 0 \le x \le l \end{cases} \quad -②$$

When $x = 0$

$$② \Rightarrow \frac{l}{2} + \frac{2l}{\pi^2} \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n^2} = l$$

$$\frac{2l}{\pi^2} \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n^2} = l - \frac{l}{2} = \frac{l}{2}$$

$$\sum_{n=1}^{\infty} \frac{1-(-1)^n}{n^2} = \frac{l}{2} \times \frac{\pi^2}{2l} = \frac{\pi^2}{4}$$

$$\Rightarrow \frac{2}{1^2} + \frac{2}{3^2} + \frac{2}{5^2} + \dots\dots = \frac{\pi^2}{4}$$

$$\Rightarrow \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} = \dots\dots \frac{\pi^2}{8} \quad //$$

④ Find the fourier Series of $f(x) = |x|$ in the interval $(-l, l)$

$$f(x) = |x| = \begin{cases} -x, & -l \le x \le 0 \\ x, & 0 \le x \le l \end{cases}$$

$$\phi(x) = -x \quad , \quad \psi(x) = x$$

$$\phi(-x) = -(-x) = x = \psi(x) \Rightarrow b_n = 0$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) \quad -①$$

$$a_0 = \frac{2}{l} \int_0^l f(x)\, dx$$

$$= \frac{2}{l} \int_0^l x\, dx$$

$$= \frac{2}{l} \left(\frac{x^2}{2}\right)_0^l$$

$$= \frac{2}{l} \left(\frac{l^2}{2}\right)$$

$$a_0 = l$$

$$a_n = \frac{2}{l} \int_0^l f(x) \cos\left(\frac{n\pi x}{l}\right) dx$$

$$= \frac{2}{l} \int_0^l x \cos\left(\frac{n\pi x}{l}\right) dx$$

$$= \frac{2}{l}\left\{ x \int_0^l \cos\left(\frac{n\pi x}{l}\right) - \int_0^l [1 \int \cos\left(\frac{n\pi x}{l}\right)dx] dx \right\}$$

$$= \frac{2}{l}\left\{ \frac{x}{n\pi}\left[x \sin\left(\frac{n\pi x}{l}\right)\right]_0^l + \frac{l^2}{n^2\pi^2}\left[\cos\left(\frac{n\pi x}{l}\right)\right]_0^l \right.$$

$$= \frac{2l^2}{l n^2\pi^2}[\cos n\pi - 1]$$

$$= \frac{2l^2}{l n^2\pi^2}[(-1)^n - 1]$$

$$a_n = \frac{2l}{n^2\pi^2}[(-1)^n - 1]$$

$$(1) \Rightarrow \frac{l}{2} + \frac{2l}{\pi^2}\sum_{n=1}^{\infty}[(-1)^n - 1]$$

$$\frac{l}{2} + \frac{2l}{n^2\pi^2}\sum_{n=1}^{\infty}[(-1)^n - 1]\cos\left(\frac{n\pi x}{l}\right) = \begin{cases} -x, & -l \le x \le 0 \\ x, & 0 \le x \le l \end{cases}$$

where $x = 0$

$$\frac{l}{2} + \frac{2l}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{n^2}\right] = 0$$

$$\sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{n^2}\right] = \frac{-l}{2} \times \frac{\pi^2}{2l}$$

$$\left[\frac{-2}{1^2}\right] + \left(\frac{-2}{3^2}\right) + \left[\frac{-2}{5^2}\right] + \cdots = \frac{-\pi^2}{4}$$

$$-2\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right] = -\frac{\pi^2}{4}$$

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8} \quad //$$

(5) Find the fourier series of $f(x) = \begin{cases} 1 + \frac{4x}{3}, & -\frac{3}{2} \le x \le 0 \\ 1 - \frac{4x}{3}, & 0 \le x \le \frac{3}{2} \end{cases}$ and hence deduce that the series $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$

$$f(x) = \begin{cases} 1 + \frac{4x}{3}, & -\frac{3}{2} \le x \le 0 \\ 1 - \frac{4x}{3}, & 0 \le x \le \frac{3}{2} \end{cases} = \begin{cases} \phi(x), & -l \le x \le 0 \\ \psi(x), & 0 \le x \le l \end{cases}$$

$$\therefore \phi(x) = 1 + \frac{4x}{3} \qquad \psi(x) = 1 - \frac{4x}{3}, \quad l = \frac{3}{2}$$

$\phi(-x) = 1 - \frac{4x}{3} = \psi(x)$

$f(x)$ is an even function, $\Rightarrow b_n = 0$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) \quad \text{—}①$$

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx$$

$$= \frac{4}{3}\int_0^{3/2} f(x)\,dx$$

$$= \frac{4}{3}\int_0^{3/2}\left(1-\frac{4x}{3}\right)dx$$

$$= \frac{4}{3}\left[x - \frac{2x^2}{3}\right]_0^{3/2}$$

$$= \frac{4}{3}\left[\frac{3}{2} - \frac{2}{3}\left(\frac{3}{2}\right)^2\right]$$

$$= \frac{4}{3}\left[\frac{3}{2} - \frac{3}{2}\right]$$

$$a_0 = 0$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{4}{3}\int_0^{3/2}\left(1-\frac{4x}{3}\right)\cos\left(\frac{2n\pi x}{3}\right)dx$$

$$= \frac{4}{3}\left\{\left(1-\frac{4x}{3}\right)\int_0^{3/2}\cos\left(\frac{2n\pi x}{3}\right)dx - \int_0^{3/2}\left[\left(\frac{-4}{3}\right)\int\cos\left(\frac{2n\pi x}{3}\right)dx\right]dx\right\}$$

$$= \frac{4}{3}\left\{\frac{3}{2n\pi}\left[\left(1-\frac{4x}{3}\right)\sin\left(\frac{2n\pi x}{3}\right)\right]_0^{3/2} - \frac{4}{3}\,\frac{3^2}{4n^2\pi^2}\left[\cos\left(\frac{2n\pi x}{3}\right)\right]_0^{3/2}\right\}$$

$$= \frac{4}{3}\left\{0 - \frac{3}{n^2\pi^2}\left[\cos n\pi - \cos 0\right]\right\}$$

$$= \frac{-4}{n^2\pi^2}\left[\cos n\pi - 1\right]$$

$$= \frac{-4}{n^2\pi^2}\left[(-1)^n - 1\right]$$

$$a_n = \frac{4}{n^2\pi^2}\left(1-(-1)^n\right)$$

$$① \Rightarrow \frac{4}{\pi^2}\sum_{n=1}^{\infty}\left(\frac{1-(-1)^n}{n^2}\right)\cos\left(\frac{2n\pi x}{3}\right) = \begin{cases} 1+\frac{4x}{3}, & -\frac{3}{2} \le x \le 0 \\ 1-\frac{4x}{3}, & 0 \le x \le 3/2 \end{cases} \quad \text{—}②$$

when $x = 0$

$$② \Rightarrow \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n^2} = 1$$

$$\sum_{n=1}^{\infty} \frac{1-(-1)^n}{n^2} = \frac{\pi^2}{4}$$

$$\Rightarrow \frac{2}{1^2} + \frac{2}{3^2} + \frac{2}{5^2} + \cdots\cdots = \frac{\pi^2}{4}$$

$$\Rightarrow \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\cdots = \frac{\pi^2}{8} //$$

⑥ Find the half range cosine series of $f(x) = x(l-x)$ in the interval $[0, l]$

Given $f(x) = x(l-x)$, $x \in [0, l]$

WKT The fourier half range cosine series in $[0, l]$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) \quad -①$$

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx$$

$$= \frac{2}{l}\int_0^l (lx - x^2)\,dx$$

$$= \frac{2}{l}\left[\frac{lx^2}{2} - \frac{x^3}{3}\right]_0^l$$

$$= \frac{2}{l}\left[\frac{l^3}{2} - \frac{l^3}{3}\right]$$

$$a_0 = \frac{2}{l} \times \frac{l^3}{6}$$

$$a_0 = \frac{l^2}{3}$$

$$a_n = \frac{2}{l}\int_0^l a_n \cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{l}\int_0^l (lx - x^2)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{l}\left\{(lx - x^2)\int_0^l \cos\left(\frac{n\pi x}{l}\right)dx - \int_0^l \left[(l-2x)\int \cos\left(\frac{n\pi x}{l}\right)dx\right]dx\right\}$$

$$= \frac{2}{l}\left\{\frac{l}{n\pi}(0-0) - \frac{l}{n\pi}\int_0^l (l-2x)\sin\left(\frac{n\pi x}{l}\right)dx\right\}$$

$$= \frac{-2}{n\pi}\left\{(l-2x)\int_0^l \sin\left(\frac{n\pi x}{l}\right)dx - \int_0^l\left[(-2)\int \sin\left(\frac{n\pi x}{l}\right)dx\right]dx\right\}$$

$$= \frac{-2}{n\pi}\left\{\frac{-l}{n\pi}\left[(l-2x)\cos\left(\frac{n\pi x}{l}\right)\right]_0^l - \frac{2l^2}{n^2\pi^2}\left(\sin\left(\frac{n\pi x}{l}\right)\right)_0^l\right\}$$

$$= \frac{-2}{n\pi}\left\{\frac{-l}{n\pi}\left[-l\cos n\pi - l\right] - 0\right\}$$

$$a_n = \frac{-2l^2}{n^2\pi^2}\left[(-1)^n + 1\right]$$

$$\therefore f(x) = \frac{l^2}{6} + \left(-\frac{2l^2}{\pi^2}\right)\sum_{n=1}^{\infty}\left(\frac{(-1)^n+1}{n^2}\right)\cos\left(\frac{n\pi x}{l}\right) //$$

(7) Find the half range cosine series $f(x) = (x-1)^2$ in the interval $[0,1]$.

Given $f(x) = (x-1)^2, \; x \in [0,1]$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) \quad \text{—(1)}$$

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx$$

$$= \frac{2}{l}\int_0^1 (x-1)^2\,dx$$

$$= 2\left[\frac{(x-1)^3}{3}\right]_0^1$$

$$a_0 = \frac{2}{3}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= 2\int_0^1 f(x)\cos(n\pi x)\,dx$$

$$= 2\int_0^1 (x-1)^2\cos(n\pi x)\,dx$$

$$= 2\left\{(x-1)^2\int_0^1 \cos(n\pi x)\,dx - \int_0^1\left[2(x-1)\int\cos(n\pi x)\,dx\right]dx\right\}$$

$$= 2\left\{\frac{1}{n\pi}(x-1)^2\left[\sin(n\pi x)\right]_0^1 + \frac{2}{n^2\pi^2}\left[\cos(n\pi x)\,2(x-1)\right]_0^1\right\}$$

$$a_n = \frac{4}{n^2\pi^2}\left\{ (x-1)\cos(n\pi x)] + 0 \right\}$$

$$= \frac{4}{n^2\pi^2}\left\{ [0+1] \right\}$$

$$a_n = \frac{4}{n^2\pi^2}$$

$$f(x) = \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{n^2}\cos(n\pi x) \quad //$$

(8) Find the half range cosine series of $f(x) = (x+1)$ in the interval $0 \le x \le 1$

$$f(x) = (x+1),\ x \in [0,1]$$

$$f(x) = (x+1),\ x \in [0,1]$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) - (1)$$

$$a_0 = \frac{2}{l}\int_0^1 (x+1)\,dx$$

$$= 2\left[\frac{x^2}{2} + x\right]_0^1$$

$$= 2\left[\frac{1}{2} + 1\right]$$

$$= 2\left[\frac{1+2}{2}\right]$$

$$= 2\left[\frac{3}{2}\right]$$

$$a_0 = 3$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cdot\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{1}\int_0^1 (x+1)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= 2\left\{ (x+1)\int_0^1 \cos\left(\frac{n\pi x}{l}\right)dx - \int_0^1 \left[(1)\int \cos\left(\frac{n\pi x}{l}\right)dx\right]dx \right\}$$

$$= 2\left\{ \frac{x+1}{n\pi}\cos(n\pi x)\Big]_0^1 - \int_0^1 \sin\left(\frac{n\pi x}{l}\right)dx \right\} dx$$

$$= \frac{2}{n\pi}\left\{ (2\cos(n\pi) - 1) - 0 \right\}$$

$$a_n = \frac{4}{n\pi}\left[(-1)^n - 1\right]$$

$$f(x) = \frac{3}{2} + \sum_{n=1}^{\infty} \frac{4}{n\pi}\left[(-1)^n - 1\right]\cos\left(\frac{n\pi x}{l}\right)$$

$$f(x) = \frac{3}{2} + \frac{4}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{n}\right]\cos\left(\frac{n\pi x}{l}\right)$$

(9) Find the half range sin series of e^x in the interval $0 \le x \le 1$

$$f(x) = e^x \quad x \in [0, 1]$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) dx$$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{l}\int_0^l e^x \sin\left(\frac{n\pi x}{l}\right)dx$$

$$= 2\int_0^1 e^x \sin(n\pi x)\,dx$$

$$b_n = 2\left[\frac{e^x}{1+n^2\pi^2}\left(\sin(n\pi x) - n\pi\cos(n\pi x)\right)\right]_0^1$$

$$= 2\left[\frac{e}{1+n^2\pi^2}\left[\sin(n\pi) - n\pi\cos(n\pi) - \frac{e^0}{1+n^2\pi^2}\left[\sin(n\pi 0) - n\pi\right.\right.\right.$$

$$\cos(n\pi$$

$$= 2\left[\frac{e}{1+n^2\pi^2}\left[-n\pi\cos(n\pi) - \frac{1}{1+n^2\pi^2}(-n\pi)\right]\right.$$

$$b_n = 2\left[\frac{e}{1+n^2\pi^2}\left(-n\pi(-1)^n\right] + \frac{n\pi}{1+n^2\pi^2}\right]$$

$$= \frac{2n\pi}{1+n^2\pi^2}\left[-(-1)^n e^1 + 1\right]$$

$$b_n = \frac{2n\pi}{1+n^2\pi^2}\left[1-(-1)e^1\right]$$

$$① \Rightarrow f(x) = 2\pi \sum_{n=1}^{\infty} \frac{n}{1+n^2\pi^2}\left[1-(-1)e^1) \sin(n\pi x)\right]$$

Harmonic Fourier Series :-

① Let $y = f(x)$ be a periodic function of the period 2π, then the fourier series of $f(x)$ in the harmonics can be expressed as $f(x) = \frac{a_0}{2} + (a_1\cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x) + \ldots\ldots$

where $\frac{a_0}{2}$ is called the constant term. (a_1, b_1) are called the first harmonic co-efficients, (a_2, b_2) are called the second harmonic coefficients and which will be evaluated as

$$a_0 = \frac{2}{N}\sum y$$

$$a_1 = \frac{2}{N}\sum y\cos x$$

$$b_1 = \frac{2}{N}\sum y \sin x$$

$$a_2 = \frac{2}{N}\sum y\cos 2x$$

$$b_2 = \frac{2}{N}\sum y\sin 2x$$

Generally, $a_n = \frac{2}{N}\sum y \cos nx, \ \forall n = 1, 2, 3, \ldots\ldots$

$$b_n = \frac{2}{N}\sum y \sin x, \ \forall n = 1, 2, 3 \ldots\ldots$$

where N is the number of terms given in the table which should always be even number.

(2) Suppose $y = f(x)$ be a periodic function over the period $2l$, then $f(x) = \frac{a_0}{2} + \left[a_1 \cos\left(\frac{\pi x}{l}\right) + b_1 \sin\left(\frac{\pi x}{l}\right)\right] + \left[a_2 \cos\left(\frac{2\pi x}{l}\right) + b_2 \sin\left(\frac{2\pi x}{l}\right)\right] + \ldots\ldots$

where $a_0 = \frac{2}{N} \sum y$ $\quad \therefore \theta = \frac{\pi x}{l}$

$$a_1 = \frac{2}{N} \sum y \cos\theta$$

$$b_1 = \frac{2}{N} \sum y \sin\theta$$

$$a_n = \frac{2}{N} \sum y \cos n\theta$$

$$b_n = \frac{2}{N} \sum y \sin n\theta.$$

(1) The following value of function y gives the displacement in inches of a certain machine part for rotation x of a flywheel. Expand y interms of fourier series upto the second harmonic.

rotations	x	0	$\pi/6$	$2\pi/6$	$3\pi/6$	$4\pi/6$	$5\pi/6$	π
displacement	y	0	9.2	14.4	17.8	17.3	11.7	0

The fourier series of $y = f(x)$ upto the second harmonic is

x	y	$y\cos x$	$y\sin x$	$y\cos 2x$	$y\sin 2x$
$0°$	0	0	0	0	0
$30°$	9.2	7.9674	4.6000	4.6	7.9674
$60°$	14.4	7.2	12.4707	-7.2	12.4707
$90°$	17.8	0	17.8	-17.8	0
$120°$	17.3	-8.65	14.9822	-8.65	-14.9822
$150°$	11.7	-10.1325	5.85	5.85	-10.1325
Σ	70.4	-3.6151	55.7029	-23.20	-4.6766

$N = 6$

$\therefore a_0 = \frac{2}{N}\sum y = \frac{2}{6}(70.4) = 23.46$

$\therefore a_1 = \frac{2}{N}\sum y\cos x = \frac{2}{6}(-3.6151) = 1.2050$

$\therefore b_1 = \frac{2}{N}\sum y\sin x = \frac{2}{6}(55.7029) = 18.5676$

$a_2 = \frac{2}{N}\sum y\cos 2x = \frac{2}{6}(-23.20) = -7.7333$

$b_2 = \frac{2}{N}\sum y\sin 2x = \frac{2}{6}(-4.6766) = -1.5589$

$f(x) = \frac{a_0}{2} + (a_1\cos x + b_1\sin x) + (a_2\cos 2x + b_2\sin 2x)$

$= \frac{23.46}{2} + (-1.2050\cos x + 18.5676\sin x) + (-7.7333\cos 2x -$

$1.5589\sin 2x)$

$= 11.73 + (-1.2050\cos x + 18.5676\sin x) + (-7.7333\cos 2x - 1.558$

$-9\sin 2x)$ //

(2) Express y as a fourier series upto first harmonic for the given data

x	0	30	60	90	120	150	180	210	240	270	300	330
y	1.8	1.1	0.30	0.16	1.50	1.30	2.16	1.25	1.30	1.52	1.76	2.00

x	y	$y\cos x$	$y\sin x$
0	1.8	1.8	0
30	1.1	0.9526	0.55
60	0.30	0.15	0.2598
90	0.16	0	0.16
120	1.50	-0.75	1.2990
150	1.30	-1.1258	0.65
180	2.16	-2.16	0
210	1.25	-1.0825	-0.625
240	1.30	-0.65	-1.1258
270	1.52	0	-1.52
300	1.76	0.88	-1.5242
330	2.00	1.7320	-1
Σ	16.15	-0.2537	-2.8762

$$\therefore a_0 = \frac{2}{N}\Sigma y = \frac{2}{12}(16.15) = 2.6916$$

$$a_1 = \frac{2}{N}\Sigma y\cos x = \frac{2}{12}(-0.2537) = -0.0422$$

$$b_1 = \frac{2}{N}\Sigma y\sin x = \frac{2}{12}(-2.8762) = -0.47936$$

$$f(x) = \frac{2.6916}{2} + (-0.0422\cos x - 0.47936\sin x)$$

$$= 1.3458 + (-0.0422\cos x - 0.47936\sin x)$$ //

③ Compute the first two harmonic of the fourier series for the given data

x°	0	60°	120°	180°	240°	300°
y	7.9	7.2	3.6	0.5	0.9	0.8

x	y	$y\cos x$	$y\sin x$	$y\cos 2x$	$y\sin 2x$
0	7.9	7.9	0	7.9	0
60	7.2	3.6	6.2353	-3.6	6.2353
120	3.6	-1.8	3.1176	-1.8	-3.1176
180	0.5	-0.5	0	0.5	0
240	0.9	-0.45	-0.7794	-0.45	0.7794
300	0.8	0.4	-0.6928	-0.4	-0.69282
Σ	20.9	9.15	7.8807	2.15	3.20428

$N = 6$

$$a_0 = \frac{2}{N}\Sigma y = \frac{2}{6}(20.9) = 6.96666$$

$$a_1 = \frac{2}{N}\Sigma y\cos x = \frac{2}{6}(9.15) = 3.05$$

$$b_1 = \frac{2}{N}\Sigma y\sin x = \frac{2}{6}(7.8807) = 2.6269$$

$$a_2 = \frac{2}{N}\Sigma y\cos 2x = \frac{2}{6}(2.15) = 71.6666$$

$$b_2 = \frac{2}{N}\Sigma y\sin 2x = \frac{2}{6}(3.20428) = 1.06809$$

$$f(x) = \frac{a_0}{2} + (a_1\cos x + b_1\sin x) + a_2\cos 2x + b_2\sin 2x)$$

$$= \frac{6.96666}{2} + (3.05\cos x + 2.6269\sin x) + (71.6666\cos 2x + 1.06809\sin 2x)$$

$$= 3.48333 + (3.05\cos x + 2.6269\sin x) + (71.666\cos 2x + 1.06809\sin 2x) //$$

(4) The displacement y (in cm) of a machine part occurs due to the rotation of x radians in given below:-

Rotation x (in radians)	0	π/3	2π/3	π	4π/3	5π/3	2π
Displacement y (in cms)	1.0	1.4	1.9	1.7	1.5	1.2	1.0

Expand y in terms of fourier series upto second harmonics.

x	y	$y\cos x$	$y\sin x$	$y\cos 2x$	$y\sin 2x$
0	1.0	1	0	1	0
60	1.4	0.7	1.2124	−0.7	1.2124
120	1.9	−0.95	1.6454	−0.95	−1.6454
180	1.7	−1.7	0	1.7	0
240	1.5	−0.75	−1.2990	−0.75	1.2990
300	1.2	0.6	−1.03923	−0.6	−1.03923
Σ	8.7	−1.1	0.51957	−0.3	−0.17323

$N = 6$

$$a_0 = \frac{2}{N}\sum y = \frac{2}{6}(8.7) = 2.9$$

$$a_1 = \frac{2}{N}\sum y\cos x = \frac{2}{6}(-1.1) = -0.36666$$

$$b_1 = \frac{2}{N}\sum y\sin x = \frac{2}{6}(0.51957) = 0.17319$$

$$a_2 = \frac{2}{N}\sum y\cos 2x = \frac{2}{6}(-0.3) = -0.1$$

$$b_2 = \frac{2}{N}\sum y\sin 2x = \frac{2}{6}(-0.17323) = -0.05774$$

$$f(x) = \frac{a_0}{2} + (a_1\cos x + b_1\sin x) + (a_2\cos 2x + b_2\sin 2x)$$

$$= \frac{2.9}{2} + (-0.36666\cos x + 0.17319\sin x) + (-0.1\cos 2x) - 0.05774\sin 2x)$$

$$= 1.45 + (-0.3666\cos x + 0.17319\sin x) + (-0.1\cos 2x - 0.05774\sin 2x).$$

(5) In a machine the displacement y of a given point in given for a certain angle x as follows.

x	0	30	60	90	120	150	180	210	240	270	300	330
y	7.9	8	7.2	5.6	3.6	1.7	0.5	0.2	0.9	2.5	4.7	6.8

Find the constant term and the first two harmonics in fourier series expansion of y.

x	y	$y\cos x$	$y\sin x$	$y\cos 2x$	$y\sin 2x$
0	7.9	7.9	0	7.9	0
30	8	6.9282	4	4	6.9282
60	7.2	3.6	6.2353	-3.6	6.2353
90	5.6	0	5.6	-5.6	0
120	3.6	-1.8	3.1176	-1.8	-3.1176
150	1.7	-1.4722	0.85	0.85	-1.4722
180	0.5	-0.5	0	0.5	0
210	0.2	-0.17320	-0.1	0.1	0.17320
240	0.9	-0.45	-0.779422	-0.45	0.77942
270	2.5	0	-2.5	-2.5	0
300	4.7	2.35	-4.0703	-2.34	-4.07031
330	6.8	5.8889	-3.4	3.4	-5.8889
Σ	49.6	22.2717	-7.45	0.45	-0.43289

$N = 12$

$a_0 = \frac{2}{N}\Sigma y = \frac{2}{12}(49.6) = 8.26666$

$a_1 = \frac{2}{N}\Sigma y\cos x = \frac{2}{12}(22.2717) = 3.71195$

$b_1 = \frac{2}{N}\Sigma y\sin x = \frac{2}{12}(-7.45) = -1.24166$

$a_2 = \frac{2}{N}\Sigma y\cos 2x = \frac{2}{12}(0.45) = 0.075$

$b_2 = \frac{2}{N}\Sigma y\sin 2x = \frac{2}{12}(-0.43289) = -0.072148$

$f(x) = \frac{a_0}{2} + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x)$

$= \frac{8.22666}{2} + (3.71195 \cos x - 1.24166 \sin x) + (0.075 \cos 2x - 0.072148 \sin 2x)$

$= 4.13333 + (3.71195 \cos x - 1.24166 \sin x) + (0.075 \cos 2x - 0.072148 \sin 2x)$

⑥ Obtain the constant term and first sin, cosine terms in the fourier expansion of y from the following table.

x	0	1	2	3	4	5
y	4	8	15	7	6	2

The given x varies as $0 \le x < 6$

$2l = 6$

$l = 3$

The fourier series expansion upto first harmonic is defined as $f(x) = \frac{a_0}{2} + a_1 \cos\left(\frac{\pi x}{3}\right) + b_1 \sin\left(\frac{\pi x}{3}\right)$ —①

x	y	$\theta = \frac{\pi x}{3}$	$y \cos\theta$	$y \sin\theta$
0	4	0	4	0
1	8	$60°$	4	6.9282
2	15	$120°$	-7.5	12.9903
3	7	$180°$	-7	0
4	6	$240°$	3	-5.1961
5	2	$300°$	1	-1.7320
Σ	42	—	-8.5	12.9901

$\therefore a_0 = \frac{2}{N} \Sigma y = \frac{2}{6}(42) = 14$

$a_1 = \frac{2}{N} \Sigma y \cos\theta = \frac{2}{6}(-8.5) = -2.8333$

$b_1 = \frac{2}{N} \Sigma y \sin\theta = \frac{2}{6}(12.9901) = 4.3301$

$\therefore f(x) = 7 - (2.8333)\cos\left(\frac{\pi x}{3}\right) + (4.3301)\sin\left(\frac{\pi x}{3}\right)$ //

(7) obtain the constant term and first sine cosine terms in the fourier expansion of y from the following table.

x	0	1	2	3	4	5
y	9	18	24	28	26	20

The given x varies $0 \le x < 6$

$2l = 6$

$l = 3$

The fourier series expansion upto the first harmonic is defined as

$$f(x) = \frac{a_0}{2} + a_1\cos\left(\frac{\pi x}{3}\right) + b_1\sin\left(\frac{\pi x}{3}\right) \quad -① $$

x	y	$\theta = \frac{\pi x}{3}$	$y\cos\theta$	$y\sin\theta$
0	9	0	9	0
1	18	60°	9	15.5884
2	24	120°	-12	20.7846
3	28	180°	-28	0
4	26	240°	-13	-22.5166
5	20	300°	10	-17.3205
Σ	125	–	-25	-3.4641

$\therefore a_0 = \frac{2}{N}\Sigma y = \frac{2}{6}(125) = 41.667$

$a_1 = \frac{2}{N}\Sigma y\cos\theta = \frac{2}{6}(-25) = -8.3333$

$b_1 = \frac{2}{N}\Sigma y\sin\theta = \frac{2}{6}(-3.4641) = -1.547$

$f(x) = \frac{41.667}{2} - 8.3333\cos\left(\frac{\pi x}{3}\right) - 1.547\sin\left(\frac{\pi x}{3}\right)$

$= 20.8335 - 8.3333\cos\left(\frac{\pi x}{3}\right) - 1.547\sin\left(\frac{\pi x}{3}\right)$ //

(8) The following table gives the variations of periodic current over a period T. show by numerical analysis that there is a direct current part 0.75 Amp. The Variable current and obtain the amplitude of first harmonic.

t (sec)	0	T/6	T/3	T/2	2T/3	5T/6	T
A	1.98	1.30	1.05	1.30	-0.88	-0.25	1.98

Given the period to a circuit $2l = T$

$l = \frac{T}{2}$

t	A	$\theta = \frac{\pi t}{l} = \frac{2\pi t}{T}$	$A\cos\theta$	$A\sin\theta$
0	1.98	0°	1.98	0
T/6	1.30	60°	0.65	1.1258
T/3	1.05	120°	-0.5250	0.9093
T/2	1.30	180°	-1.30	0
2T/3	-0.88	240°	0.44	0.7621
5T/6	-0.25	300°	-0.1250	0.2165

$$\therefore a_0 = \frac{2}{N}\sum A = \frac{2}{6}(4.5) = 1.5$$

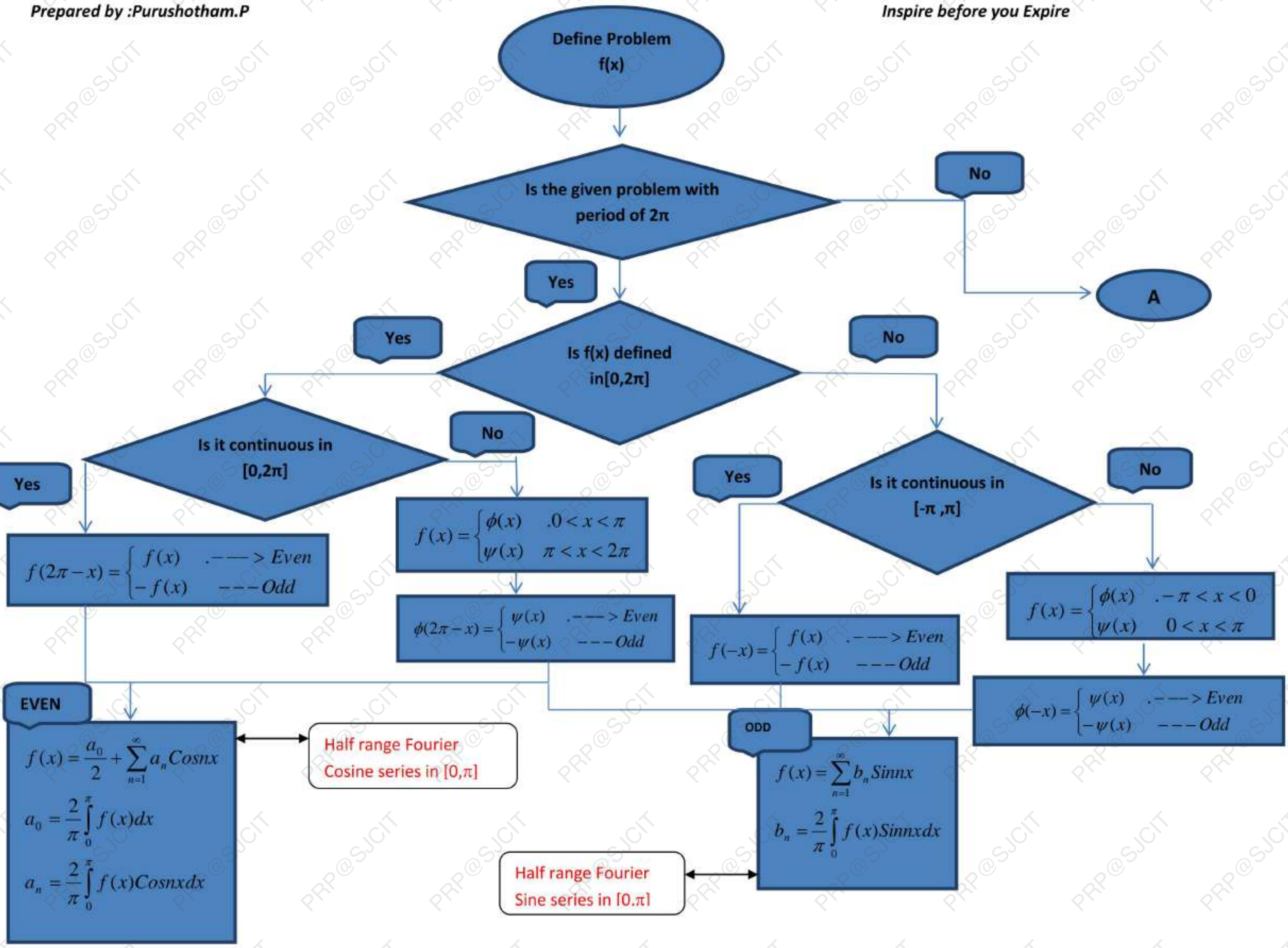
$$a_1 = \frac{2}{N}\sum A\cos\theta = \frac{2}{6}(1.12) = 0.3733$$

$$b_1 = \frac{2}{N}\sum A\sin\theta = \frac{2}{6}(3.0137) = 1.0045$$

$$\text{Direct current} = \frac{a_0}{2} = \frac{1.5}{2} = 0.75 \text{ Amp}$$

$$\text{Amplitude} = \sqrt{a_1^2 + b_1^2} = \sqrt{(0.3733)^2 + (1.0045)^2}$$

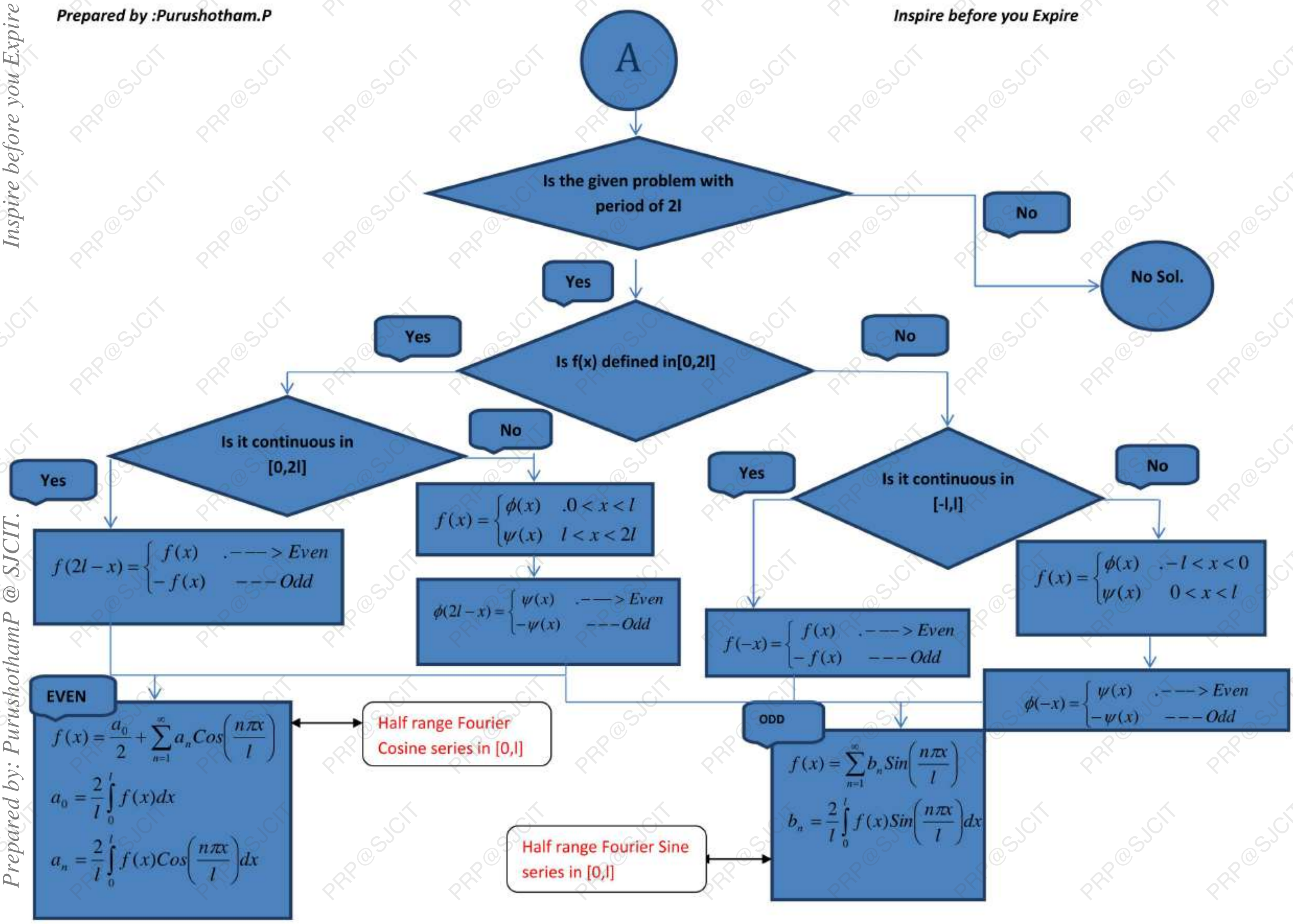
$$= 1.0716 //$$



Define Problem f(x)
Is the given problem with period of 2π
No
A
Yes
Is f(x) defined in[0,2π]
Yes
No
Is it continuous in [0,2π]
Yes
No
$f(2\pi - x) = \begin{cases} f(x) & .---> Even \\ -f(x) & ---Odd \end{cases}$
$f(x) = \begin{cases} \phi(x) & 0 < x < \pi \\ \psi(x) & \pi < x < 2\pi \end{cases}$
$\phi(2\pi - x) = \begin{cases} \psi(x) & .---> Even \\ -\psi(x) & ---Odd \end{cases}$
Is it continuous in [-π ,π]
Yes
No
$f(-x) = \begin{cases} f(x) & .---> Even \\ -f(x) & ---Odd \end{cases}$
$f(x) = \begin{cases} \phi(x) & -\pi < x < 0 \\ \psi(x) & 0 < x < \pi \end{cases}$
$\phi(-x) = \begin{cases} \psi(x) & .---> Even \\ -\psi(x) & ---Odd \end{cases}$
EVEN
$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n Cosnx$
$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx$
$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)Cosnxdx$
Half range Fourier Cosine series in [0,π]
ODD
$f(x) = \sum_{n=1}^{\infty} b_n Sinnx$
$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)Sinnxdx$
Half range Fourier Sine series in [0.π]

Prepared by :Purushotham.P

Inspire before you Expire



Unit - IV

Vector differentiation and Vector operators

Vector Differentiation :-

Derivative :- Let $\bar{f}$ be a vector function on an interval I and $a \in I$. Then $\lim_{t \to a} \frac{\bar{f}(t) - \bar{f}(a)}{t-a}$, if exists, is called derivative of $\bar{f}$ at a. It is denoted by $\bar{f}'(a)$ or $\left[\frac{d\bar{f}}{dt}\right]$ at $t = a$.

we now state some properties of differentiable functions.

1) Derivative of a constant vector is $\bar{0}$

If $\bar{a}$ & $\bar{b}$ are differentiable vector functions, then

2) $\frac{d}{dt}(\bar{a} \pm \bar{b}) = \frac{d\bar{a}}{dt} \pm \frac{d\bar{b}}{dt}$

3). $\frac{d}{dt}(\bar{a} \cdot \bar{b}) = \frac{d\bar{a}}{dt} \cdot \bar{b} + \bar{a} \cdot \frac{d\bar{b}}{dt}$

4). $\frac{d}{dt}(\bar{a} \times \bar{b}) = \frac{d\bar{a}}{dt} \times \bar{b} + \bar{a} \times \frac{d\bar{b}}{dt}$.

5) If $\bar{f}$ is a differentiable vector function and ϕ is a scalar differentiable function. Then $\frac{d}{dt}(\phi \bar{f}) = \phi \frac{d\bar{f}}{dt} + \frac{d\phi}{dt} \bar{f}$.

6) If $\bar{f} = f_1(t)\bar{i} + f_2(t)\bar{j} + f_3(t)\bar{k}$ where $f_1(t), f_2(t), f_3(t)$ are Cartesian components of the vector $\bar{f}$, then $\frac{d\bar{f}}{dt} = \frac{df_1}{dt}\bar{i} + \frac{df_2}{dt}\bar{j} + \frac{df_3}{dt}\bar{k}$.

7). The necessary & sufficient condition for $\bar{f}(t)$ to be constant vector function is $\frac{d\bar{f}}{dt} = \bar{0}$.

Partial derivatives Properties :-

1). $\frac{\partial}{\partial t}(\phi \bar{a}) = \frac{\partial \phi}{\partial t} \bar{a} + \phi \frac{\partial \bar{a}}{\partial t}$

2) If λ is a constant, then $\frac{\partial}{\partial t}(\lambda \bar{a}) = \lambda \frac{\partial \bar{a}}{\partial t}$

3). If $\bar{c}$ is a constant vector, then $\frac{\partial}{\partial t}(\phi \bar{c}) = \bar{c} \frac{\partial \phi}{\partial t}$

4). $\frac{\partial}{\partial t}(\bar{a} \pm \bar{b}) = \frac{\partial \bar{a}}{\partial t} \pm \frac{\partial \bar{b}}{\partial t}$

5). $\frac{\partial}{\partial t}(\bar{a} \cdot \bar{b}) = \frac{\partial \bar{a}}{\partial t} \cdot \bar{b} + \bar{a} \cdot \frac{\partial \bar{b}}{\partial t}$

6) $\frac{\partial}{\partial t}(\bar{a} \times \bar{b}) = \frac{\partial \bar{a}}{\partial t} \times \bar{b} + \bar{a} \times \frac{\partial \bar{b}}{\partial t}$

7) Let $\bar{f} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$ where f_1, f_2, f_3 are differential scalar functions of more than one variable. Then $\frac{\partial \bar{f}}{\partial t} = \bar{i} \frac{\partial f_1}{\partial t} + \bar{j} \frac{\partial f_2}{\partial t} + \bar{k} \cdot \frac{\partial f_3}{\partial t}$

Vector differential operator :-

The vector differential operator ∇ (read as nabla or del) is defined as $\nabla = \bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}$.

we will define some functions known as "Gradient", "divergence", "curl" involving this operator ∇.

Gradient of a Scalar Point function :-

Let $\phi(x, y, z)$ be a scalar point function. Then the vector function $\bar{i}\frac{\partial \phi}{\partial x} + \bar{j}\frac{\partial \phi}{\partial y} + \bar{k}\frac{\partial \phi}{\partial z}$ is known as the gradient of ϕ.

It is denoted by grad ϕ (or) $\nabla\phi$.

$$\nabla\phi = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)\phi = \bar{i}\frac{\partial \phi}{\partial x} + \bar{j}\frac{\partial \phi}{\partial y} + \bar{k}\frac{\partial \phi}{\partial z}.$$

Properties :-

1) If f and g are two scalar functions. Then grad $(f \pm g)$ = grad $f \pm$ grad g.
2) The neccessory & sufficient condition for a scalar point function to be constant is that $\nabla f = \bar{0}$.
3) grad $(fg) = f$ (grad g) + g (grad f)
4) If c is a constant, grad $(cf) = c$ (grad f)
5) $\text{grad}\left(\frac{f}{g}\right) = \frac{g(\text{grad} f) - f(\text{grad} g)}{g^2}$, $(g \neq 0)$.
6) Let $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$. Then $d\bar{r} = (dx)\bar{i} + (dy)\bar{j} + (dz)\bar{k}$.

   If ϕ is any scalar point function, Then $d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz$

$$d\phi = \left(\bar{i}\frac{\partial \phi}{\partial x} + \bar{j}\frac{\partial \phi}{\partial y} + \bar{k}\frac{\partial \phi}{\partial z}\right) \cdot (\bar{i}dx + \bar{j}dy + \bar{k}dz)$$

$$= \nabla\phi \cdot d\bar{r}.$$

Directional Derivative :-

The directional derivative of a scalar point function ϕ at point "P" in the direction of unit vector $\bar{e}$ is equal to $\bar{e} \cdot \text{grad}\phi = \bar{e} \cdot \nabla\phi$.

Unit vector :-

$\bar{e}$ be the unit vector (Normal) in the direction of $\bar{a}$. Then

$$\bar{e} = \frac{\bar{a}}{|\bar{a}|}.$$

1) Find $\nabla(x^2+y^2z)$

Sol:- Let $f(x,y,z)=x^2+y^2z$. Then $\bar{i}\frac{\partial}{\partial x}f+\bar{j}\frac{\partial}{\partial y}f+\bar{k}\frac{\partial}{\partial z}f$

$$\nabla f = \bar{i}\frac{\partial}{\partial x}(x^2+y^2z)+\bar{j}\frac{\partial}{\partial y}(x^2+y^2z)+\bar{k}\frac{\partial}{\partial z}(x^2+y^2z)$$

$$\nabla f = \bar{i}(2x)+\bar{j}(2yz)+\bar{k}(y^2) = 2x\bar{i}+2yz\bar{j}+y^2\bar{k}$$

Note:- A vector function of $\bar{r}=x\bar{i}+y\bar{j}+z\bar{k}$ is called a Position vector function.

$r=|\bar{r}|=\sqrt{x^2+y^2+z^2}$, $r^2=x^2+y^2+z^2$.

Diff Partially w.r.to "x", "y", "z" we have

$\cancel{2}r.\frac{\partial r}{\partial x}=\cancel{2}x \Rightarrow \frac{\partial r}{\partial x}=\frac{x}{r}$, $2r.\frac{\partial r}{\partial y}=2y \Rightarrow \frac{\partial r}{\partial y}=\frac{y}{r}$

$2r.\frac{\partial r}{\partial z}=2z \Rightarrow \frac{\partial r}{\partial z}=\frac{z}{r}$

2). Prove that $\nabla(r^n)=nr^{n-2}.\bar{r}$.

Sol:- Let $\bar{r}=x\bar{i}+y\bar{j}+z\bar{k}$. Let $r=|\bar{r}|$. Then we have $r^2=x^2+y^2+z^2$

Diff Partially w.r.to "x" we have

$\frac{\partial r}{\partial x}=\frac{x}{r}$, $\frac{\partial r}{\partial y}=\frac{y}{r}$, $\frac{\partial r}{\partial z}=\frac{z}{r}$

$$\therefore \nabla(r^n)=\bar{i}\frac{\partial}{\partial x}(r^n)+\bar{j}\frac{\partial}{\partial y}(r^n)+\bar{k}\frac{\partial}{\partial z}(r^n)=\Sigma\bar{i}\frac{\partial}{\partial x}(r^n).$$

$$=\Sigma\bar{i}\,nr^{n-1}\frac{\partial r}{\partial x}$$

$$=\Sigma\bar{i}\,nr^{n-1}.\frac{x}{r} = \Sigma\bar{i}\,nr^{n-1-1}.x = nr^{n-2}.\Sigma\bar{i}x$$

$$=nr^{n-2}.\bar{r}$$

$$\boxed{\therefore \nabla(r^n)=nr^{n-2}.\bar{r}.}$$

3). $\nabla\left(\frac{1}{r}\right)$.

Sol:- we have $\nabla(r^n)=nr^{n-2}.\bar{r}$.

Given that $\nabla\left(\frac{1}{r}\right)=\nabla(r^{-1})$

put $n=-1$

$$\nabla(r^{-1})=(-1)r^{-1-2}.\bar{r}$$

$$=-\bar{r}.r^{-3}$$

$$=\frac{-\bar{r}}{r^3}$$

$$\therefore \boxed{\nabla\left(\frac{1}{r}\right)=\frac{-\bar{r}}{r^3}}$$

4). Find grad (r).

Sol:- Grad$(r)=\nabla(r)$.

we have $\nabla(r^n)=nr^{n-2}\bar{r}$

where $n=1$

$$\nabla(r)=1r^{1-2}.\bar{r}$$

$$=r^{-1}.\bar{r}$$

$$=\frac{\bar{r}}{r}$$

$$\boxed{\therefore \nabla(r)=\frac{\bar{r}}{r}}$$

5). If $\bar{r}$ is the Position vector of the Point $P(x,y,z)$. Then Prove that $\nabla f(r) = f'(r)\cdot\frac{\bar{r}}{r}$

Sol:- Let $\bar{r} = x\bar{i}+y\bar{j}+z\bar{k}$ and let $r = |\bar{r}|$. Then we have $r^2 = x^2+y^2+z^2$.

Diff partially w.r.to "x" & "y" & "z" respectively, we get

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}.$$

$$\nabla[f(r)] = \left[\bar{i}\frac{\partial}{\partial x}+\bar{j}\frac{\partial}{\partial y}+\bar{k}\frac{\partial}{\partial z}\right]f(r) = \bar{i}f'(r)\frac{\partial r}{\partial x}+\bar{j}f'(r)\frac{\partial r}{\partial y}+\bar{k}\cdot f'(r)\frac{\partial r}{\partial z}$$

$$= \Sigma\,\bar{i}\,f'(r)\cdot\frac{x}{r}$$

$$= \Sigma\bar{i}f'(r)\cdot\frac{x}{r} = \frac{f'(r)}{r}\cdot\Sigma\bar{i}x = f'(r)\cdot\frac{\bar{r}}{r}$$

6). Find the directional derivative of $2xy+z^2$ at $(1,-1,3)$ in the direction of $\bar{i}+2\bar{j}+3\bar{k}$.

Sol:- Let $f = 2xy+z^2$, $\bar{a} = \bar{i}+2\bar{j}+3\bar{k}$.

Directional derivative of "f" in the direction of $\bar{a}$ is $\nabla f\cdot\frac{\bar{a}}{|\bar{a}|}$

To find $\nabla f = \bar{i}\frac{\partial f}{\partial x}+\bar{j}\frac{\partial f}{\partial y}+\bar{k}\cdot\frac{\partial f}{\partial z}$

$$\frac{\partial f}{\partial x} = 2y, \quad \frac{\partial f}{\partial y} = 2x, \quad \frac{\partial f}{\partial z} = 2z$$

$\therefore \nabla f = 2y\bar{i}+2x\bar{j}+2z\bar{k}$.

$(\text{grad} f)$ at $(1,-1,3) = 2(-1)\bar{i}+2(1)\bar{j}+2(3)\bar{k} = -2\bar{i}+2\bar{j}+6\bar{k}$.

Given vector is $\bar{a} = \bar{i}+2\bar{j}+3\bar{k} \Rightarrow |\bar{a}| = \sqrt{1+4+9} = \sqrt{14}$.

$\therefore$ D.D of 'f' in the direction of $\bar{a}$ is $\frac{\bar{a}\cdot\nabla f}{|\bar{a}|} = \frac{(\bar{i}+2\bar{j}+3\bar{k})\cdot(-2\bar{i}+2\bar{j}+6\bar{k})}{\sqrt{14}}$

$$= \frac{-2+4+18}{\sqrt{14}} = \frac{20}{\sqrt{14}}$$

7). Find the directional derivative of $\phi = x^2yz+4xz^2$ at $(1,-2,-1)$ in the direction $2\bar{i}-\bar{j}-2\bar{k}$.

Sol:- Given $\phi = x^2yz+4xz^2$. $\bar{a} = 2\bar{i}-\bar{j}-2\bar{k}$ $\Rightarrow |\bar{a}| = \sqrt{4+1+4} = \sqrt{9} = 3$.

$$\frac{\partial\phi}{\partial x} = 2xyz+4z^2, \quad \frac{\partial\phi}{\partial y} = x^2z, \quad \frac{\partial\phi}{\partial z} = x^2y+8x.$$

$$\nabla\phi = (2xyz+4z^2)\bar{i}+(x^2z)\bar{j}+(x^2y+8x)\bar{k}.$$

$$(\nabla\phi)_{at\,(1,-2,-1)} = i(4+4)+j(-1)+k(-2-8) = 8i-j-10k.$$

Directional derivative $= \nabla\phi\cdot\frac{\bar{a}}{|\bar{a}|} = \frac{(8i-j-10k)(2\bar{i}-\bar{j}-2\bar{k})}{3}$

$$= \frac{16+1+20}{3} = \frac{37}{3}$$

8) Find the directional derivative of $f(x,y,z) = xy^2 + yz^3$ at the Point IV-3.i

$(2,-1,1)$ in the direction of the vector $i + 2j + 2k$.

Sol:- Given $f(x,y,z) = xy^2 + yz^3$, $\frac{\partial f}{\partial x} = y^2$, $\frac{\partial f}{\partial y} = 2xy + z^3$, $\frac{\partial f}{\partial z} = 3yz^2$

Hence $\nabla f = y^2 i + (2xy + z^3) j + 3yz^2 k$.

∇f at $(2,-1,1) = i - 3j - 3k$.

The unit vector in the direction of the vector $i + 2j + 2k$ is $\bar{e} = \frac{i+2j+2k}{\sqrt{1^2+2^2+2^2}}$

$\therefore \bar{e} = \frac{1}{3}(i + 2j + 2k)$

Required directional derivative $= \nabla f \cdot \bar{e} = \frac{1}{3}(i+2j+2k)(i-3j-3k)$

$= \frac{1}{3}(1-6-6) = \frac{-11}{3}$.

9). Find the directional derivative of the function $f = x^2 - y^2 + 2z^2$ at the Point $P = (1,2,3)$ in the direction of the line PQ where $Q = (5,0,4)$.

Sol:- The Position vectors of P and Q with respect to the origin are

$\overline{OP} = \bar{i} + 2\bar{j} + 3\bar{k}$ and $\overline{OQ} = 5\bar{i} + 4\bar{k}$

$\therefore \overline{PQ} = \overline{OQ} - \overline{OP} = 4\bar{i} - 2\bar{j} + \bar{k}$

Let $\bar{e}$ be the unit vector in the direction of $\overline{PQ}$. Then $\bar{e} = \frac{4\bar{i} - 2\bar{j} + \bar{k}}{\sqrt{21}}$.

$\text{grad} f = \bar{i}\frac{\partial f}{\partial x} + \bar{j}\frac{\partial f}{\partial y} + \bar{k}\frac{\partial f}{\partial z} = 2x\bar{i} - 2y\bar{j} + 4z\bar{k}$

$\therefore$ The directional derivative of f at $P(1,2,3)$ in the direction $\overline{PQ}$ is

$\therefore \overline{PQ} = \bar{e}\cdot\nabla f = \frac{1}{\sqrt{21}}(4\bar{i} - 2\bar{j} + \bar{k})(2x\bar{i} - 2y\bar{j} + 4z\bar{k})$

$= \frac{1}{\sqrt{21}}(8x + 4y + 4z) = \frac{1}{\sqrt{21}}(28)$

at $(1,2,3)$

10) Find the directional derivative of $\frac{1}{r}$ in the direction of $\bar{r} = xi + yj + zk$ at $(1,1,2)$.

Sol:- Let $f = \frac{1}{r}$.

Now $\text{grad} f = \nabla f = \frac{-\bar{r}}{r^3}$ (From Problem 2. $\nabla r^n = n r^{n-2}\,\bar{r}$

$n=-1$ $\nabla r^{-1} = (-1) r^{-1-2}\bar{r}$

$= -\bar{r}\cdot r^{-3}$

$= -\frac{\bar{r}}{r^3}$

$\therefore \nabla(\frac{1}{r}) = \frac{-\bar{r}}{r^3}$.

unit vector $\bar{e} = \frac{\bar{r}}{|\bar{r}|} = \frac{\bar{r}}{r}$

$\therefore |\bar{r}| = r$.

Directional derivative $= \nabla f \cdot \bar{e}$

$= \frac{-\bar{r}}{r^3}\cdot\frac{\bar{r}}{r} = \frac{-(\bar{r})^2}{r^4}$

$= -\frac{1}{r^4}(x^2+y^2+z^2)$

$= \frac{-1}{r^4}(r^2) = -\frac{1}{r^2} = \frac{-1}{(x^2+y^2+z^2)}$ at $(1,1,2)$

$= \frac{-1}{6}$

11). Find a unit normal vector to the given surface $x^2y+2z=4$ at the point $(2,-2,3)$

Sol:- Let the given surface be $f=x^2y+2z-4$. $\frac{\partial f}{\partial x}=2xy+2z$, $\frac{\partial f}{\partial y}=x^2$, $\frac{\partial f}{\partial z}=2x$.

$\therefore \text{grad} f = \bar{i}\frac{\partial f}{\partial x}+\bar{j}\frac{\partial f}{\partial y}+\bar{k}\cdot\frac{\partial f}{\partial z} = (2xy+2z)\bar{i}+x^2\bar{j}+(2x)\bar{k}$.

$(\text{grad} f)$ at $(2,-2,3) = (-8+6)\bar{i}+4\bar{j}+4\bar{k} = -2\bar{i}+4\bar{j}+4\bar{k}$.

Hence the required unit normal vector $= \frac{\nabla f}{|\nabla f|} = \frac{\cancel{2}(-\bar{i}+2\bar{j}+2\bar{k})}{\cancel{2}\sqrt{1+2^2+2^2}} = \underline{\underline{\frac{-\bar{i}+2\bar{j}+2\bar{k}}{3}}}$

12). Find the unit normal vector to the surface $z=x^2+y^2$ at $(-1,-2,5)$

Sol:- Let the given surface be $f=x^2+y^2-z$. $\frac{\partial f}{\partial x}=2x$, $\frac{\partial f}{\partial y}=2y$, $\frac{\partial f}{\partial z}=-1$.

$\text{grad} f = \bar{i}\cdot\frac{\partial f}{\partial x}+\bar{j}\frac{\partial f}{\partial y}+\bar{k}\cdot\frac{\partial f}{\partial z} = (2x)\bar{i}+(2y)\bar{j}-\bar{k}$

(∇f) at $(-1,-2,5) = -2\bar{i}-4\bar{j}-\bar{k}$

Hence the required unit normal vector $= \frac{\nabla f}{|\nabla f|} = \frac{-2\bar{i}-4\bar{j}-\bar{k}}{\sqrt{(-2)^2+(-4)^2+(1)^2}} = \underline{\underline{\frac{-2\bar{i}-4\bar{j}-\bar{k}}{\sqrt{21}}}}$

13). Find the constants a and b, so that the surface $ax^2-byz=(a+2)x$ will be orthogonal to the surface $4x^2y+z^3=4$ at the point $(+1,-1,2)$.

Sol:- Let the given surfaces be $f(x,y,z)=ax^2-byz-(a+2)x \rightarrow (1)$

$g(x,y,z)=4x^2y+z^3-4 \rightarrow (2)$

Given the two surfaces meet at the point $(1,-1,2)$.

Sub the point in eqn (1) we get $\cancel{a}+2b-(\cancel{a}+2)=0 \Rightarrow 2b=2 \Rightarrow \boxed{b=1}$

Now $\frac{\partial f}{\partial x}=2ax-(a+2)$, $\frac{\partial f}{\partial y}=-bz$, $\frac{\partial f}{\partial z}=-by$

$\nabla f = (2ax-(a+2))\bar{i}-(bz)\bar{j}-(by)\bar{k}$.

$(\nabla f)_{(1,-1,2)} = [2a-(a+2)]\bar{i}-2b\bar{j}+bk = (a-2)\bar{i}-2b\bar{j}+b\bar{k}$.

$= (a-2)\bar{i}-2\bar{j}+(1)\bar{k} = \bar{n}_1$ normal vector to surface 1.

Also $\frac{\partial g}{\partial x}=8xy$, $\frac{\partial g}{\partial y}=4x^2$, $\frac{\partial g}{\partial z}=3z^2$.

$\nabla g = (8xy)\bar{i}+(4x^2)\bar{j}+(3z^2)\bar{k}$

$(\nabla g)_{(1,-1,2)} = -8\bar{i}+4\bar{j}+12\bar{k} = \bar{n}_2$, normal vector to surface 2.

Given the surfaces $f(x,y,z)$, $g(x,y,z)$ are orthogonal at the point $(1,-1,2)$

$[\nabla f]\cdot[\nabla g]=0 \Rightarrow [(a-2)\bar{i}-2\bar{j}+\bar{k}]\cdot[-8\bar{i}+4\bar{j}+12\bar{k}]=0$.

$-8a+16-8+12=0$

$+8a=+20 \Rightarrow a=\frac{20}{8}=\frac{5}{2}$ $\boxed{\therefore a=\frac{5}{2}}$

Hence $a=\frac{5}{2}$ and $b=1$

14). Evaluate the angle between the normals to the surface $xy = z^2$ at the points $(4,1,2)$ and $(3,3,-3)$. IV-4.i

Sol:- Given surface is $f(x,y,z) = xy - z^2 \rightarrow (1)$

Let $\bar{n}_1$ and $\bar{n}_2$ be the normals to this surface at $(4,1,2)$ & $(3,3,-3)$ respectively

Diff eqn (1) partially w.r.to "x", "y", "z" respectively.

$$\frac{\partial f}{\partial x} = y, \quad \frac{\partial f}{\partial y} = x, \quad \frac{\partial f}{\partial z} = -2z.$$

$$\therefore \text{grad } f = y\bar{i} + x\bar{j} - 2z\bar{k}.$$

$$\bar{n}_1 = (\text{grad} f) \text{ at } (4,1,2) = \bar{i} + 4\bar{j} - 4\bar{k}$$

$$\bar{n}_2 = (\text{grad} f) \text{ at } (3,3,-3) = 3\bar{i} + 3\bar{j} + 6\bar{k}.$$

Let "θ" be the angle between the two normals

$$\cos\theta = \frac{\bar{n}_1 \cdot \bar{n}_2}{|\bar{n}_1| \cdot |\bar{n}_2|} = \frac{(\bar{i}+4\bar{j}-4\bar{k})(3\bar{i}+3\bar{j}+6\bar{k})}{\sqrt{1+4^2+4^2}\cdot\sqrt{3^2+3^2+6^2}} = \frac{3+12-24}{\sqrt{33}\cdot\sqrt{54}} = \frac{-9}{\sqrt{33}\cdot\sqrt{54}}.$$

15) Find the angle between the surfaces $x^2+y^2+z^2=9$ and $z = x^2+y^2-3$ at the point $(2,-1,2)$.

Sol:- Let $\phi_1 = x^2+y^2+z^2-9=0$ and $\phi_2 = x^2+y^2-z-3=0$ be the given surfaces.

Then $\nabla\phi_1 = 2x\bar{i} + 2y\bar{j} + 2z\bar{k}$ and $\nabla\phi_2 = 2x\bar{i} + 2y\bar{j} - \bar{k}$.

Let $\bar{n}_1$ and $\bar{n}_2$ be the two normals to this surface at $(2,-1,2)$

Let $\bar{n}_1 = \nabla\phi_1$ at $(2,-1,2) = 4\bar{i} - 2\bar{j} + 4\bar{k}$.

$\bar{n}_2 = \nabla\phi_2$ at $(2,-1,2) = 4\bar{i} - 2\bar{j} - \bar{k}$.

Let "θ" be the angle between the two normals surfaces.

$$\cos\theta = \frac{\bar{n}_1 \cdot \bar{n}_2}{|\bar{n}_1| \cdot |\bar{n}_2|} = \frac{(4\bar{i}-2\bar{j}+4\bar{k})(4\bar{i}-2\bar{j}-\bar{k})}{\sqrt{4^2+(-2)^2+(4)^2}\cdot\sqrt{4^2+(-2)^2+(-1)^2}} = \frac{16+4-4}{6\sqrt{21}} = \frac{16}{6\sqrt{21}} = \frac{8}{3\sqrt{21}}.$$

$$\therefore \cos\theta = \frac{8}{3\sqrt{21}} \Rightarrow \theta = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right)$$

Divergence of a vector :-

Let $\bar{f}$ be any continuously differentiable vector point function. Then $\bar{i}\cdot\frac{\partial \bar{f}}{\partial x}+\bar{j}\cdot\frac{\partial \bar{f}}{\partial y}+\bar{k}\cdot\frac{\partial \bar{f}}{\partial z}$ is called the divergence of $\bar{f}$. It is denoted by $\operatorname{div}\bar{f}$.

i.e., $\operatorname{div}\bar{f}=\bar{i}\cdot\frac{\partial \bar{f}}{\partial x}+\bar{j}\cdot\frac{\partial \bar{f}}{\partial y}+\bar{k}\cdot\frac{\partial \bar{f}}{\partial z}=\left(\bar{i}\frac{\partial}{\partial x}+\bar{j}\frac{\partial}{\partial y}+\bar{k}\frac{\partial}{\partial z}\right)\cdot\bar{f}$.

Hence we can write $\operatorname{div}\bar{f}$ as $\operatorname{div}\bar{f}=\nabla\cdot\bar{f}$.

If $\bar{f}=f_1\bar{i}+f_2\bar{j}+f_3\bar{k}$

$$\operatorname{div}\bar{f}=\left(\bar{i}\frac{\partial}{\partial x}+\bar{j}\frac{\partial}{\partial y}+\bar{k}\cdot\frac{\partial}{\partial y}\right)\cdot\left(f_1\bar{i}+f_2\bar{j}+f_3\bar{k}\right)$$

$$=\frac{\partial f_1}{\partial x}+\frac{\partial f_2}{\partial y}+\frac{\partial f_3}{\partial z}.$$

Note :-

1). $\nabla\cdot\bar{f}\neq\bar{f}\cdot\nabla$

2). $\operatorname{div}\bar{f}=\Sigma\,\bar{i}\cdot\frac{\partial \bar{f}}{\partial x}$ if $\bar{f}=f_1\bar{i}+f_2\bar{j}+f_3\bar{k}$ and $\operatorname{div}\bar{f}=\Sigma\frac{\partial f_1}{\partial x}$

Solenoidal vector :-

A vector point function $\bar{f}$ is said to be solenoidal if $\boxed{\operatorname{div}\bar{f}=0}$

1). If $\bar{f}=xy^2\bar{i}+2x^2yz\bar{j}-3yz^2\bar{k}$. find $\operatorname{div}\bar{f}$ at $(1,-1,1)$.

Sol:- Given $\bar{f}=xy^2\bar{i}+2x^2yz\bar{j}-3yz^2\bar{k}$.

$$\operatorname{div}\bar{f}=\frac{\partial f_1}{\partial x}+\frac{\partial f_2}{\partial y}+\frac{\partial f_3}{\partial z}=\frac{\partial}{\partial x}(xy^2)+\frac{\partial}{\partial y}(2x^2yz)+\frac{\partial}{\partial z}(-3yz^2)$$

$$\operatorname{div}\bar{f}=y^2+2x^2z-6yz$$

$$\therefore(\operatorname{div}\bar{f})\text{ at }(1,-1,1)=1+2+6=9.$$

2). If $\bar{f}=(x+3y)\bar{i}+(y-2z)\bar{j}+(x+Pz)\bar{k}$ is solenoidal, find P.

Sol:- Let $\bar{f}=(x+3y)\bar{i}+(y-2z)\bar{j}+(x+Pz)\bar{k}=f_1\bar{i}+f_2\bar{j}+f_3\bar{k}$.

we have $\operatorname{div}\bar{f}=\frac{\partial f_1}{\partial x}+\frac{\partial f_2}{\partial y}+\frac{\partial f_3}{\partial z}=\frac{\partial}{\partial x}(x+3y)+\frac{\partial}{\partial y}(y-2z)+\frac{\partial}{\partial z}(x+Pz)$

$\therefore\operatorname{div}\bar{f}=1+1+P=2+P$.

$\therefore$ Since $\bar{f}$ is solenoidal, we have $\operatorname{div}\bar{f}=0\Rightarrow 2+P=0\Rightarrow\boxed{P=-2}$

3). Find $\operatorname{div}\bar{r}$ where $\bar{r}=x\bar{i}+y\bar{j}+z\bar{k}$.

Sol:- we have $\bar{r}=x\bar{i}+y\bar{j}+z\bar{k}=f_1\bar{i}+f_2\bar{j}+f_3\bar{k}$.

$$\operatorname{div}\bar{r}=\frac{\partial f_1}{\partial x}+\frac{\partial f_2}{\partial y}+\frac{\partial f_3}{\partial z}=\frac{\partial}{\partial x}(x)+\frac{\partial}{\partial y}(y)+\frac{\partial}{\partial z}(z)=1+1+1=3$$

4) Find $div\,\bar{f}$ when $\bar{f} = grad(x^3+y^3+z^3-3xyz)$.

Sol:- Let $\phi = x^3+y^3+z^3-3xyz$. Then

$$\frac{\partial\phi}{\partial x} = 3x^2-3yz, \quad \frac{\partial\phi}{\partial y} = 3y^2-3xz, \quad \frac{\partial\phi}{\partial z} = 3z^2-3xy.$$

$$grad\,\phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\cdot\frac{\partial\phi}{\partial z} = (3x^2-3yz)\bar{i} + (3y^2-3xz)\bar{j} + (3z^2-3xy)\bar{k}.$$

Hence $div\,\bar{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} = \frac{\partial}{\partial x}(3x^2-3yz) + \frac{\partial}{\partial y}(3y^2-3xz) + \frac{\partial}{\partial z}(3z^2-3xy)$

$$div\,\bar{f} = 6x+6y+6z = 6(x+y+z)$$

Imp
5). Find $div\,\bar{f}$ where $\bar{f} = r^n.\bar{r}$. Find n if it is solenoidal (or)

Prove that $r^n\bar{r}$ is solenoidal if $n=-3$. (or)

Prove that $div(r^n\bar{r}) = (n+3)r^n$. Hence show that $\bar{r}/r^3$ is solenoidal.

Sol:- Given $\bar{f} = r^n\bar{r}$ where $\bar{r} = x\bar{i}+y\bar{j}+z\bar{k}$ and $r = |\bar{r}|$.

we have $r^2 = x^2+y^2+z^2$

Diff Partially w.r.to "x".

$$2r.\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}, \quad \text{Simillarlly, } \frac{\partial r}{\partial y} = \frac{y}{r} \text{ and } \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\therefore \bar{f} = r^n\bar{r} = r^n(x\bar{i}+y\bar{j}+z\bar{k}) = r^nx\bar{i}+r^ny\bar{j}+r^nz\bar{k} = f_1\bar{i}+f_2\bar{j}+f_3\bar{k}$$

$$\therefore div\,\bar{f} = \frac{\partial f_1}{\partial x}+\frac{\partial f_2}{\partial y}+\frac{\partial f_3}{\partial z} = \frac{\partial}{\partial x}(r^nx)+\frac{\partial}{\partial y}(r^ny)+\frac{\partial}{\partial z}(r^nz). \qquad \frac{\partial}{\partial x}(u.v)$$

$$= nr^{n-1}\frac{\partial r}{\partial x}.x + nr^{n-1}\frac{\partial r}{\partial y}.y + nr^{n-1}.\frac{\partial r}{\partial z}.z + r^n+r^n+r^n$$

$$= nr^{n-1}.\frac{x}{r}.x + nr^{n-1}.\frac{y}{r}.y + nr^{n-1}.\frac{z}{r}.z + 3r^n$$

$$= nr^{n-1}\left[\frac{x^2}{r}+\frac{y^2}{r}+\frac{z^2}{r}\right]+3r^n = nr^{n-1}.\frac{(r^{\cancel{2}})}{\cancel{r}}+3r^n = nr^{n-1+1}+3r^n$$

$$= nr^n+3r^n = (n+3)r^n.$$

Let $\bar{f} = r^n.\bar{r}$ be solenoidal. Then $div\,\bar{f} = 0$.

$$\therefore (n+3)r^n = 0 \Rightarrow n+3=0 \Rightarrow \boxed{n=-3}$$

6). Evaluate $\nabla.\left(\frac{\bar{r}}{r^3}\right)$ where $\bar{r} = x\bar{i}+y\bar{j}+z\bar{k}$ and $r=|\bar{r}|$.

(or)

show that $\frac{\bar{r}}{r^3}$ is solenoidal.

Sol:- we have $\bar{r} = x\bar{i}+y\bar{j}+z\bar{k}$ and $r = |\bar{r}| = \sqrt{x^2+y^2+z^2}$

and $\frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}.$

$$\text{Now } \frac{\bar{r}}{r^3} = \bar{r}r^{-3} = r^{-3}(x\bar{i}+y\bar{j}+z\bar{k}) = r^{-3}x\bar{i}+r^{-3}y\bar{j}+r^{-3}z\bar{k}$$
$$= f_1\bar{i}+f_2\bar{j}+f_3\bar{k}.$$

Hence $\nabla\cdot\left(\frac{\bar{r}}{r^3}\right) = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$.

we have $f_1 = r^{-3}x \Rightarrow \frac{\partial f_1}{\partial x} = r^{-3}.1 + (-3)r^{-4}x.\frac{\partial r}{\partial x}$

$= r^{-3} - 3xr^{-4}.\frac{x}{r}$

$= r^{-3} - 3x^2r^{-5}$.

Simillarlly $\frac{\partial f_2}{\partial y} = r^{-3} - 3y^2r^{-5}$

$\frac{\partial f_3}{\partial z} = r^{-3} - 3z^2r^{-5}$

$\therefore \nabla\left(\frac{\bar{r}}{r^3}\right) = r^{-3} - 3x^2r^{-5} + r^{-3} - 3y^2r^{-5} + r^{-3} - 3z^2r^{-5}$

$= 3r^{-3} - 3r^{-5}(x^2+y^2+z^2) = +3r^{-3} - 3r^{-5}(r^2)$

$= 3r^{-3} - 3r^{-3} = 0$.

$\therefore$ Hence $\nabla\left(\frac{\bar{r}}{r^3}\right) = 0$

$\therefore \nabla\left(\frac{\bar{r}}{r^3}\right)$ is Solenoidal.

7). If $\bar{F} = y(ax^2+z)\bar{i} + x(y^2-z^2)\bar{j} + 2xy(z-xy)\bar{k}$ is solenoidal then find a.

Sol:- $\bar{F} = y(ax^2+z)\bar{i} + x(y^2-z^2)\bar{j} + 2xy(z-xy)\bar{k} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$.

$\text{div}\,\bar{F} = 0$ is called Solenoidal.

$\text{div}\,\bar{F} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} = \frac{\partial}{\partial x}\left[yax^2 + zy\right] + \frac{\partial}{\partial y}\left[xy^2 - xz^2\right] + \frac{\partial}{\partial z}\left[2xyz - 2x^2y^2\right]$

$= 2axy + 2yx + 2xy$

$= 2xy(a+1+1)$

$\therefore \text{div}\,\bar{F} = 0 \Rightarrow 2xy(a+2) = 0$

$a+2=0$

$\boxed{\therefore a = -2}$

Curl of a vector :-

Let $\bar{f}$ be a vector point function. Then the vector function defined by $\bar{i} \times \frac{\partial \bar{f}}{\partial x} + \bar{j} \times \frac{\partial \bar{f}}{\partial y} + \bar{k} \times \frac{\partial \bar{f}}{\partial z}$ is called curl $\bar{f}$. It is denoted by $\nabla \times \bar{f}$.

$$\therefore \text{curl } \bar{f} = \bar{i} \times \frac{\partial \bar{f}}{\partial x} + \bar{j} \times \frac{\partial \bar{f}}{\partial y} + \bar{k} \times \frac{\partial \bar{f}}{\partial z} = \Sigma\left(\bar{i} \times \frac{\partial \bar{f}}{\partial x}\right).$$

If $\bar{f} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$

$$\text{curl } \bar{f} = \nabla \times \bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$= \bar{i}\left[\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right] - \bar{j}\left(\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z}\right) + \bar{k}\left(\frac{\partial f_2}{\partial x} + \frac{\partial f_1}{\partial y}\right)$$

$$= \bar{i}\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) + \bar{j}\left(\frac{\partial f_1}{\partial z} + \frac{\partial f_3}{\partial x}\right) + \bar{k}\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)$$

Note:- 1) If $\bar{f}$ is a constant vector then curl $\bar{f} = 0$.

2). curl $(\bar{a} \pm \bar{b})$ = curl $\bar{a} \pm$ curl $\bar{b}$.

Irrotational Vector :-

A vector $\bar{f}$ is said to be irrotational vector if curl $\bar{f} = \bar{0}$.

Note:- If curl $\bar{f} = \bar{0}$ iff, there Exist a scalar function $\bar{f} = \nabla\phi$

1) Find curl $\bar{f}$ where $\bar{f} = \text{grad}(x^3 + y^3 + z^3 - 3xyz)$.

Sol:- Let $\phi = x^3 + y^3 + z^3 - 3xyz$.

$$\bar{f} = \text{grad}\,\phi = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)(x^3 + y^3 + z^3 - 3xyz)$$

$$= (3x^2 - 3yz)\bar{i} + 3(y^2 - zx)\bar{j} + 3(z^2 - xy)\bar{k}$$

$$\text{curl } \bar{f} = \text{curl grad}\,\phi$$

$$= \nabla \times \text{grad}\,\phi = 3\begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - yz & y^2 - zx & z^2 - xy \end{vmatrix}$$

$$= 3\left[\bar{i}(-x + x) - j(-y + y) + k(-z + z)\right]$$

$$= 3(0) = \bar{0}.$$

Hence curl $\bar{f} = \bar{0}$.

Note:- curl (grad ϕ) $= \bar{0}$. (i.e.,) grad ϕ is always irrotational

2). If $\bar{F} = (x+y+1)\bar{i} + \bar{j} - (x+y)\bar{k}$, Then show that $\bar{F}\cdot \text{Curl}\,\bar{F} = 0$. IV-6.ii

Sol:- Given $\bar{F} = (x+y+1)\bar{i} + \bar{j} - (x+y)\bar{k}$

$$\text{Curl}\,\bar{F} = \nabla\times\bar{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x+y+1 & 1 & -x-y \end{vmatrix} = i(-1-0) - j(-1-0) + k(0-1)$$
$$= -i + j - k.$$

$$\therefore \bar{F}\cdot\text{Curl}\,\bar{F} = \left[(x+y+1)\bar{i} + \bar{j} - (x+y)\bar{k}\right]\left[-\bar{i}+\bar{j}-\bar{k}\right] = -x-y-1+1+x+y = 0$$

3). Prove that $\text{Curl}\,\bar{r} = \bar{0}$.

Sol:- Let $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$.

$$\text{Curl}\,\bar{r} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \bar{i}(0-0) - \bar{j}(0-0) + \bar{k}(0-0)$$
$$= \bar{0}.$$

$\therefore \bar{r}$ is irrotational vector.

4). Prove that if $\bar{r}$ is the position vector of any point in space, then $r^n\bar{r}$ is irrotational. (or) show that $\text{Curl}\,(r^n\bar{r}) = 0$.

Sol:- Let $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ and $r = |\bar{r}|$ $\therefore r^2 = x^2+y^2+z^2$.

Diff w.r.to "x" Partially, we get $2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$.

Similarly $\frac{\partial r}{\partial y} = \frac{y}{r}$, $\frac{\partial r}{\partial z} = \frac{z}{r}$.

we have $r^n\bar{r} = r^n(x\bar{i} + y\bar{j} + z\bar{k})$

$$\nabla\times(r^n\bar{r}) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xr^n & yr^n & zr^n \end{vmatrix}$$

$$= \bar{i}\left\{\frac{\partial}{\partial y}(zr^n) - \frac{\partial}{\partial z}(yr^n)\right\} - \bar{j}\left\{\frac{\partial}{\partial x}(zr^n) - \frac{\partial}{\partial z}(xr^n)\right\} + \bar{k}\left\{\frac{\partial}{\partial x}(yr^n) + \frac{\partial}{\partial y}(xr^n)\right\}$$

$$= \Sigma\,\bar{i}\left\{znr^{n-1}\frac{\partial r}{\partial y} - ynr^{n-1}\frac{\partial r}{\partial z}\right\}$$

$$= \Sigma\, nr^{n-1}\,\bar{i}\left\{z\left(\frac{y}{r}\right) - y\left(\frac{z}{r}\right)\right\} = nr^{n-1}\Sigma\bar{i}\left[\left(\frac{zy}{r}\right) - \left(\frac{yz}{r}\right)\right]$$

$$= nr^{n-2}\,\Sigma\bar{i}\,(zy-yz)$$

$$= nr^{n-2}\left[(zy-yz)\bar{i} + (xz-zx)\bar{j} + (xy-yx)\bar{k}\right]$$

$$= nr^{n-2}\left[0\bar{i} + 0\bar{j} + 0\bar{k}\right] = nr^{n-2}(\bar{0})$$

$$= \bar{0}.$$

$\therefore$ Hence $r^n\bar{r}$ is irrotational

Prove that $\bar{F} = yz\bar{i} + zx\bar{j} + yx\bar{k}$ is irrotational. IV-7.i

Sol:- We have $\bar{F} = yz\bar{i} + zx\bar{j} + xy\bar{k}$.

$$\text{curl } \bar{F} = \nabla \times \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & zx & xy \end{vmatrix} = \bar{i}\left[\frac{\partial}{\partial y}(xy) - \frac{\partial}{\partial z}(zx)\right] - \bar{j}\left[\frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial z}(yz)\right] + \bar{k}\left[\frac{\partial}{\partial x}(zx) - \frac{\partial}{\partial y}(yz)\right]$$

$$= \bar{i}(x-x) - \bar{j}(y-y) + \bar{k}(z-z) = \bar{0}.$$

Hence $\bar{F}$ is irrotational

5). Find constants a, b and c if the vector

$\bar{f} = (2x+3y+az)\bar{i} + (bx+2y+3z)\bar{j} + (2x+cy+3z)\bar{k}$ is irrotational.

Sol:-

$$\text{curl } \bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x+3y+az & bx+2y+3z & 2x+cy+3z \end{vmatrix}$$

$$= \bar{i}(c-3) - \bar{j}(2-a) + \bar{k}(b-3).$$

If the vector is irrotational Then curl $\bar{f} = \bar{0}$.

$\therefore$ $c-3=0$, $2-a=0$, $b-3=0$ $\Rightarrow$ $\boxed{c=3,\ a=2,\ b=3}$

6). Show that the vector $(x^2-yz)\bar{i} + (y^2-zx)\bar{j} + (z^2-xy)\bar{k}$ is irrotational find it's scalar Potential.

Sol:- Let $\bar{f} = (x^2-yz)\bar{i} + (y^2-zx)\bar{j} + (z^2-xy)\bar{k}$

$$\text{curl } \bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2-yz & y^2-zx & z^2-xy \end{vmatrix} = i(-x+x) - j(-y+y) + k(-z+z)$$

$$= \bar{0}.$$

$\therefore$ $\bar{f}$ is irrotational. Then there exist ϕ such that $\bar{f} = \nabla\phi$.

$$\Rightarrow \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z} = (x^2-yz)\bar{i} + (y^2-zx)\bar{j} + (z^2-xy)\bar{k}.$$

Now comparing on B/S.

$$\frac{\partial\phi}{\partial x} = x^2-yz, \quad \frac{\partial\phi}{\partial y} = y^2-zx, \quad \frac{\partial\phi}{\partial z} = z^2-xy.$$

Now integrating w.r.to x, y, z respectively.

$$\int \frac{\partial\phi}{\partial x}dx = \int x^2 dx - yz\,dx + \text{constant}$$

$$\phi = \frac{x^3}{3} - xyz + \text{Constant} \rightarrow (1)$$

Simillarlly $\phi = \frac{y^3}{3} - xyz + \text{Constant} \rightarrow (2)$

$$\phi = \frac{z^3}{3} - xyz + \text{Constant} \rightarrow (3).$$

From (1) & (2) & (3).

$$\therefore \phi = \frac{x^3}{3} + \frac{y^3}{3} + \frac{z^3}{3} - xyz + \text{Constant}$$

which is the required scalar Potential

8) Find constants a, b, c so that the vector
$\bar{A} = (x+2y+az)\bar{i} + (bx-3y-z)\bar{j} + (4x+cy+2z)\bar{k}$ is irrotational.
Also find ϕ such that $\bar{A} = \nabla\phi$.

Sol:- Given vector is $\bar{A} = (x+2y+az)\bar{i} + (bx-3y-z)\bar{j} + (4x+cy+2z)\bar{k}$ → (1)

∴ vector $\bar{A}$ is irrotational ⇒ curl $\bar{A} = \bar{0}$

$$\begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x+2y+az) & (bx-3y-z) & 4x+cy+2z \end{vmatrix} = \bar{0}$$

$\bar{i}(c+1) + \bar{j}(a-4) + \bar{k}(b-2) = \bar{0}$

$\therefore \bar{i}(c+1) + \bar{j}(a-4) + \bar{k}(b-2) = 0\bar{i} + 0\bar{j} + 0\bar{k}$

Comparing Both sides

$c+1=0,\ a-4=0,\ b-2=0$ $\therefore a=4,\ b=2,\ c=-1$

Sub these values in eqn (1)

$\bar{A} = (x+2y+4z)\bar{i} + (2x-3y-z)\bar{j} + (4x-y+2z)\bar{k}$

we have to find scalar potential $\bar{A} = \nabla\phi$.

$(x+2y+4z)\bar{i} + (2x-3y-z)\bar{j} + (4x-y+2z)\bar{k} = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$

Comparing both sides we get

$\frac{\partial\phi}{\partial x} = x+2y+4z \Rightarrow \phi = \frac{x^2}{2} + 2xy + 4xz + f_1(y,z)$

$\frac{\partial\phi}{\partial y} = 2x-3y-z \Rightarrow \phi = 2\cdot xy - \frac{3y^2}{2} - yz + f_2(x,z)$

$\frac{\partial\phi}{\partial z} = 4x-y+2z \Rightarrow \phi = 4xz - yz + \frac{\cancel{2}z^2}{\cancel{2}} + f_3(x,y)$

Hence $\phi = \frac{x^2}{2} - \frac{3y^2}{2} + z^2 + 2xy + 4zx - yz + c$

(or) $\phi = \frac{x^2}{2} - \frac{3y^2}{2} + z^2 + 2xy - yz + 4zx + c$

9). Find whether the function $\bar{F} = (x^2-y^3)\bar{i} + (y^2-3x)\bar{j} + (z^2-xy)\bar{k}$ is irrotational and hence find scalar Potential Corresponding to it. IV - 8 i

Sol:- Given function $\bar{F} = (x^2-y^3)\bar{i} + (y^2-3x)\bar{j} + (z^2-xy)\bar{k}$

$$\nabla \times \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x^2-y^3) & (y^2-3x) & (z^2-xy) \end{vmatrix}$$

$$= i\left[(-x-0)\right] - j\left[(-y)-0\right] + k\left[(-3)-(0-3y^2)\right] = -x\bar{i} + y\bar{j} + \bar{k}(3y^2-3)$$

Since $\nabla \times \bar{F} \neq 0$, $\bar{F}$ is not irrotational.

Operators:-

1). Scalar differential operator $\bar{a}\cdot\nabla$:-

The Operator $(\bar{a}\cdot\nabla) = (\bar{a}\cdot\bar{i})\frac{\partial}{\partial x} + (\bar{a}\cdot\bar{j})\frac{\partial}{\partial y} + (\bar{a}\cdot\bar{k})\frac{\partial}{\partial z}$ is defined

Such That $(\bar{a}\cdot\nabla)\phi = (\bar{a}\cdot\bar{i})\frac{\partial\phi}{\partial x} + (\bar{a}\cdot\bar{j})\frac{\partial\phi}{\partial y} + (\bar{a}\cdot\bar{k})\frac{\partial\phi}{\partial z}$.

2) Vector differential operator $\bar{a}\times\nabla$:-

$$(\bar{a}\times\nabla)\phi = (\bar{a}\times\bar{i})\frac{\partial\phi}{\partial x} + (\bar{a}\times\bar{j})\frac{\partial\phi}{\partial y} + (\bar{a}\times\bar{k})\frac{\partial\phi}{\partial z}$$

3). Laplacian Operator ∇^2 :-

The Operator $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is called Laplacian Operator.

Note:-

1). $\nabla^2\phi = \nabla\cdot(\nabla\phi) = \text{div}(\text{grad}\,\phi)$

2) If $\nabla^2\phi = 0$ then ϕ is said to satisfy Laplacian ~~oper~~ equation. This ϕ is called a harmonic function.

Note:-

$A+B = B+A$

$A\cdot B = B\cdot A$

$A\times B = -B\times A$

$(A+B)\cdot C = A\cdot C + B\cdot C$

$(A+B)\times C = (A\times C)+(B\times C)$

$A\cdot(B\times C) = B\cdot(C\times A) = C\cdot(A\times B)$

$A\times(B\times C) = (A\cdot C)B - (A\cdot B)C$

$(A\times B)\times C = (A\cdot C)B - (B\cdot C)A$

$i\cdot i = j\cdot j = k\cdot k = 1$

$i\cdot j = i\cdot k = j\cdot i = k\cdot i = j\cdot \bar{k} = k\cdot \bar{j} = \bar{0}$

$\bar{i}\times\bar{i} = \bar{j}\times\bar{j} = \bar{k}\times\bar{k} = \bar{0}$

$\bar{i}\times\bar{j} = k$, $\bar{j}\times\bar{i} = -k$

$\bar{j}\times\bar{k} = i$, $\bar{k}\times\bar{j} = -i$

$\bar{k}\times\bar{i} = \bar{j}$, $\bar{i}\times\bar{k} = -\bar{j}$

IV - 8 ii

1). Prove that $\text{div}(\text{grad}\, r^m) = m(m+1)r^{m-2}$ (or)

$$\nabla^2(r^m) = m(m+1)r^{m-2}.$$

Sol:- Let $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ and $r = |\bar{r}| \Rightarrow r^2 = x^2+y^2+z^2$

Diff partially w.r.to "x", we get $2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$

Similarly $\frac{\partial r}{\partial y} = \frac{y}{r}$ and $\frac{\partial r}{\partial z} = \frac{z}{r}$.

Now $\text{grad}(r^m) = \bar{i}\frac{\partial}{\partial x}(r^m) + \bar{j}\frac{\partial}{\partial y}(r^m) + \bar{k}\frac{\partial (r^m)}{\partial z}$

$$= \Sigma \bar{i}\frac{\partial}{\partial r}(r^m) = \Sigma \bar{i}\, m r^{m-1}\frac{\partial r}{\partial x}$$

$$= \Sigma \bar{i}\, m r^{m-1}.\frac{x}{r} = \Sigma \bar{i}\, m r^{m-2} x$$

$$\text{grad}(r^m) = \bar{i}\, m r^{m-2}x + \bar{j}\, m r^{m-2}y + \bar{k}\, m r^{m-2}.z.$$

$$= f_1 x + f_2 y + f_3 z$$

$$\text{div}(\text{grad}\, r^m) = \frac{\partial}{\partial x}(m r^{m-2}x) + \frac{\partial}{\partial y}(m r^{m-2}y) + \frac{\partial}{\partial z}(m r^{m-2}z)$$

$$= \Sigma \frac{\partial}{\partial x}[m r^{m-2}x] = m\,\Sigma \frac{\partial}{\partial x}[r^{m-2}.x]$$

$$= m\Sigma\left[(m-2)r^{m-3}\frac{\partial r}{\partial x}x + r^{m-2}\right]$$

$$= m\Sigma\left[(m-2)r^{m-3}.\frac{x}{r}x + r^{m-2}\right] = m\Sigma\left[(m-2)r^{m-4}x^2 + r^{m-2}\right]$$

$$= m\left[(m-2)r^{m-4}\Sigma x^2 + \Sigma r^{m-2}\right]$$

$\Sigma x^2 = x^2+y^2+z^2 = r^2.$

$$= m\left[(m-2)r^{m-4}(r^2) + 3r^{m-2}\right]$$

$$= m\left[(m-2)r^{m-2} + 3r^{m-2}\right]$$

$$= m\left[(m-2+3)r^{m-2}\right]$$

$$= m\left[(m+1)r^{m-2}\right]$$

$$\therefore \nabla^2(r^m) = m(m+1)r^{m-2}.$$

Note:- $\nabla^2(r^{-1})$

$$\nabla^2(r^m) = m(m+1)r^{m-2}$$

where $m = -1$

$$= (-1)(-1+1)r^{-1+2}$$

$$= 0$$

$$\therefore \nabla^2(r^{-1}) = 0$$

2) Show that $\nabla^2[f(r)] = \frac{d^2f}{dr^2} + \frac{2}{r}\frac{df}{dr} = f''(r) + \frac{2}{r}f'(r)$ where $r = |\bar{r}|$. IV-9.i

Sol:- Given that $\nabla^2[f(r)] = \text{div}[\text{grad} f(r)]$

$$\text{grad}[f(r)] = \nabla f(r) = \bar{i}\frac{\partial}{\partial x}[f(r)] + \bar{j}\frac{\partial}{\partial y}[f(r)] + \bar{k}\frac{\partial}{\partial z}[f(r)]$$

$$= \bar{i} f'(r).\frac{\partial r}{\partial x} + \bar{j} f'(r).\frac{\partial r}{\partial y} + \bar{k} f'(r)\frac{\partial r}{\partial z}$$

$$= \bar{i} f'(r)\frac{x}{r} + \bar{j} f'(r)\frac{y}{r} + \bar{k} f'(r).\frac{z}{r} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k} \text{ (say)}.$$

$$\text{grad}[f(r)] = \Sigma \bar{i} f'(r)\frac{x}{r}$$

$$\text{div}[\text{grad} f(r)] = \nabla^2 f(r) = \nabla\cdot\nabla f(r) = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$$= \frac{\partial}{\partial x}\left[f'(r).\frac{x}{r}\right] + \frac{\partial}{\partial y}\left[f'(r).\frac{y}{r}\right] + \frac{\partial}{\partial z}\left[f'(r).\frac{z}{r}\right]$$

$$= \Sigma\frac{\partial}{\partial x}\left[f'(r).\frac{x}{r}\right] = \Sigma\frac{\partial}{\partial x}\left[\frac{f'(r)x}{r}\right]$$

$$= \Sigma\left[\frac{r.\frac{\partial}{\partial x}[f'(r).x] - f'(r)x\frac{\partial}{\partial x}(r)}{r^2}\right]$$

$$= \Sigma\left[\frac{r[f''(r)\frac{\partial r}{\partial x}x + f'(r)] - f'(r)x(\frac{x}{r})}{r^2}\right]$$

$$= \Sigma\left[\frac{\cancel{r}f''(r)\frac{x}{\cancel{r}}.x + rf'(r) - \frac{x^2 f'(r)}{r}}{r^2}\right]$$

$$= \frac{f''(r)}{r^2}\Sigma x^2 + \frac{1}{r}\Sigma f'(r) - \frac{1}{r^3}f'(r)\Sigma x^2$$

$$= \frac{f''(r)}{\cancel{r^2}}(\cancel{r^2}) + \frac{3}{r}f'(r) - \frac{1}{r^3}f'(r)(r^2)$$

$$= f''(r) + f'(r)\left[\frac{3}{r} - \frac{1}{r}\right] = f''(r) + \frac{2}{r}[f'(r)].$$

Hence the result

Vector Identities :-

1). Prove that curl (grad ϕ) = $\bar{0}$.

Sol:- Let ϕ be any scalar point function. Then grad $\phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$.

$$\therefore \text{curl}(\text{grad}\,\phi) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \end{vmatrix} = \bar{i}\left[\frac{\partial^2\phi}{\partial y\partial z} - \frac{\partial^2\phi}{\partial z\partial y}\right] - \bar{j}\left[\frac{\partial^2\phi}{\partial x\partial z} - \frac{\partial^2\phi}{\partial z\partial x}\right] + \bar{k}\left[\frac{\partial^2\phi}{\partial x\partial y} - \frac{\partial^2\phi}{\partial y\partial x}\right] = \bar{0}.$$

Note:- Since curl (grad ϕ) = $\bar{0}$, we have grad ϕ is always irrotational

2). Prove that div (curl $\bar{f}$) = 0.

Sol:- Let $\bar{f} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$.

$$\text{curl}\,\bar{f} = \nabla\times\bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix} = \bar{i}\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) - \bar{j}\left(\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z}\right) + \bar{k}\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)$$

$$= f_1\bar{i} + f_2\bar{j} + f_3\bar{k} \text{ (say).}$$

$$\text{div}(\text{curl}\,\bar{f}) = \nabla\cdot(\nabla\times\bar{f}) = \frac{\partial}{\partial x}(f_1) + \frac{\partial}{\partial y}(f_2) + \frac{\partial}{\partial z}(f_3)$$

$$= \frac{\partial}{\partial x}\left[\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right] - \frac{\partial}{\partial y}\left[\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z}\right] + \frac{\partial}{\partial z}\left[\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right]$$

$$= \frac{\partial^2 f_3}{\partial x\partial y} - \frac{\partial^2 f_2}{\partial x\partial z} - \frac{\partial^2 f_3}{\partial y\partial x} + \frac{\partial^2 f_1}{\partial y\partial z} + \frac{\partial^2 f_2}{\partial z\partial x} - \frac{\partial^2 f_1}{\partial z\partial y}$$

$$= 0.$$

Note:- Since div (curl $\bar{f}$) = 0 therefore curl $\bar{f}$ is always ~~irrotat~~ solenoidal.

3). If $\bar{a}$ is a diff function and ϕ is a diff scalar function then prove that div $(\phi\bar{a})$ = (grad ϕ)$\cdot\bar{a}$ + ϕ div $\bar{a}$ (or) $\nabla\cdot(\phi\bar{a}) = (\nabla\phi)\cdot\bar{a} + \phi(\nabla\cdot\bar{a})$.

Sol:- L.H.S. $\text{div}(\phi\bar{a}) = \nabla\cdot(\phi\bar{a})$

$$= \Sigma\,\bar{i}\cdot\frac{\partial}{\partial x}(\phi\bar{a})$$

$$= \Sigma\,\bar{i}\cdot\left(\frac{\partial\phi}{\partial x}\bar{a} + \phi\frac{\partial\bar{a}}{\partial x}\right)$$

$$= \Sigma\left(\bar{i}\cdot\frac{\partial\phi}{\partial x}\bar{a}\right) + \phi\,\Sigma\left(\bar{i}\cdot\frac{\partial\bar{a}}{\partial x}\right)$$

$$= \Sigma\left(\bar{i}\frac{\partial\phi}{\partial x}\right)\cdot\bar{a} + \phi\,\Sigma\left(\bar{i}\cdot\frac{\partial\bar{a}}{\partial x}\right)$$

$$\text{div}(\phi\bar{a}) = (\nabla\phi)\cdot\bar{a} + (\nabla\cdot\bar{a})\phi$$

$\because\ \bar{a}\cdot(l\bar{b})$
$(\bar{a}l)\cdot\bar{b}$
$l(\bar{a}\cdot\bar{b})$.

4). Prove that curl $(\phi\bar{a})$ = (grad ϕ) $\times\,\bar{a}$ + ϕ curl $\bar{a}$.

Proof:- L.H.S. $\text{curl}(\phi\bar{a}) = \Sigma\,\bar{i}\times\frac{\partial}{\partial x}(\phi\bar{a})$.

$$= \Sigma\,\bar{i}\times\left[\frac{\partial\phi}{\partial x}\bar{a} + \phi\frac{\partial\bar{a}}{\partial x}\right]$$

$$= \Sigma\left(\bar{i}\times\frac{\partial\phi}{\partial x}\bar{a}\right) + \phi\,\Sigma\left(\bar{i}\times\frac{\partial\bar{a}}{\partial x}\right)$$

$$= \Sigma\left(\bar{i}\frac{\partial\phi}{\partial x}\right)\times\bar{a} + \Sigma\left(\bar{i}\times\frac{\partial\bar{a}}{\partial x}\right)\phi$$

$$= (\nabla\phi)\times\bar{a} + (\nabla\times\bar{a})\phi$$

$$= (\text{grad}\,\phi)\times\bar{a} + \phi\,\text{curl}\,\bar{a}$$

5). Prove that $div(\bar{a}\times\bar{b}) = \bar{b}\cdot curl\,\bar{a} - \bar{a}\cdot curl\,\bar{b}$

$$\nabla\cdot(\bar{a}\times\bar{b}) = \bar{b}\cdot(\nabla\times\bar{a}) - \bar{a}\cdot(\nabla\times\bar{b})$$

Proof:- L.H.S. $div(\bar{a}\times\bar{b}) = \sum \bar{i}\cdot\frac{\partial}{\partial x}(\bar{a}\times\bar{b})$

$$= \sum \bar{i}\cdot\left[\frac{\partial\bar{a}}{\partial x}\times\bar{b} + \bar{a}\times\frac{\partial\bar{b}}{\partial x}\right]$$

$$= \sum \bar{i}\cdot\left(\frac{\partial\bar{a}}{\partial x}\times\bar{b}\right) + \sum \bar{i}\cdot\left(\bar{a}\times\frac{\partial\bar{b}}{\partial x}\right)$$

$$= \sum\left(\bar{i}\times\frac{\partial\bar{a}}{\partial x}\right)\cdot\bar{b} - \sum\left(\bar{i}\times\frac{\partial\bar{b}}{\partial x}\right)\cdot\bar{a}$$

$[A\cdot(B\times C) = B\cdot(C\times A) = C\cdot(A\times B)]$

$\left[\begin{matrix} A\cdot B = B\cdot A \\ A\times B = -B\times A\end{matrix}\right]$

$$= curl\,\bar{a}\cdot\bar{b} - curl\,\bar{b}\cdot\bar{a}$$

$$= \bar{b}\cdot curl\,\bar{a} - \bar{a}\cdot curl\,\bar{b}$$

$$= R.H.S$$

6). Prove that $grad(\bar{a}\cdot\bar{b}) = (\bar{b}\cdot\nabla)\bar{a} + (\bar{a}\cdot\nabla)\bar{b} + \bar{b}\times curl\,\bar{a} + \bar{a}\times curl\,\bar{b}$

(or)

P/T $\nabla(\bar{A}\cdot\bar{B}) = (\bar{B}\cdot\nabla)\bar{A} + (\bar{A}\cdot\nabla)\bar{B} + \bar{B}\times(\nabla\times\bar{A}) + \bar{A}\times(\nabla\times\bar{B})$.

Proof:-

~~L.H.S:-~~ consider

$$\bar{a}\times curl\,\bar{b} = \bar{a}\times(\nabla\times\bar{b}) = \bar{a}\times\left[\sum\bar{i}\times\frac{\partial\bar{b}}{\partial x}\right]$$

$$= \sum\bar{a}\times\left[\bar{i}\times\frac{\partial\bar{b}}{\partial x}\right]$$

$[\because A\times(B\times C) = (A\cdot C)B - (A\cdot B)C$ and $(A\times B)\times C = (A\cdot C)B - (B\cdot C)A]$

$$= \sum\left[\left(\bar{a}\cdot\frac{\partial\bar{b}}{\partial x}\right)\bar{i} - (\bar{a}\cdot\bar{i})\frac{\partial\bar{b}}{\partial x}\right] = \sum\bar{i}\left(\bar{a}\cdot\frac{\partial\bar{b}}{\partial x}\right) - \sum(\bar{a}\cdot\bar{i})\frac{\partial\bar{b}}{\partial x}$$

$$= \sum\bar{i}\left(\bar{a}\cdot\frac{\partial\bar{b}}{\partial x}\right) - \left(\bar{a}\cdot\sum\bar{i}\frac{\partial}{\partial x}\right)\bar{b}$$

$$\bar{a}\times curl\,\bar{b} = \sum\bar{i}\left(\bar{a}\cdot\frac{\partial\bar{b}}{\partial x}\right) - (\bar{a}\cdot\nabla)\bar{b} \rightarrow (1)$$

Simillarlly

$$\bar{b}\times curl\,\bar{a} = \sum\bar{i}\left(\bar{b}\cdot\frac{\partial\bar{a}}{\partial x}\right) - (\bar{b}\cdot\nabla)\bar{a} \rightarrow (2)$$

eqn (1) + (2) gives

$$\bar{a}\times curl\,\bar{b} + \bar{b}\times curl\,\bar{a} = \sum\bar{i}\left(\bar{a}\cdot\frac{\partial\bar{b}}{\partial x}\right) - (\bar{a}\cdot\nabla)\bar{b} + \sum\bar{i}\left(\bar{b}\cdot\frac{\partial\bar{a}}{\partial x}\right) - (\bar{b}\cdot\nabla)\bar{a}$$

$$\bar{a}\times curl\,\bar{b} + \bar{b}\times curl\,\bar{a} + (\bar{a}\cdot\nabla)\bar{b} - (\bar{b}\cdot\nabla)\bar{a} = \sum\bar{i}\left(\bar{a}\cdot\frac{\partial\bar{b}}{\partial x}\right) + \sum\bar{i}\left(\bar{b}\cdot\frac{\partial\bar{a}}{\partial x}\right)$$

$$= \sum\bar{i}\left(\bar{a}\cdot\frac{\partial\bar{b}}{\partial x} + \bar{b}\cdot\frac{\partial\bar{a}}{\partial x}\right)$$

$$= \sum\bar{i}\frac{\partial}{\partial x}(\bar{a}\cdot\bar{b})$$

$$= \nabla(\bar{a}\cdot\bar{b})$$

$$= grad(\bar{a}\cdot\bar{b})$$

Hence the theoream

7). curl $(\bar{a}\times\bar{b}) = \bar{a}\,\text{div}\,\bar{b} - \bar{b}\,\text{div}\,\bar{a} + (\bar{b}\cdot\nabla)\bar{a} - (\bar{a}\cdot\nabla)\bar{b}$.

(or)

$$\nabla\times(\bar{a}\times\bar{b}) = (\nabla\cdot\bar{b})\bar{a} - (\nabla\cdot\bar{a})\bar{b} + (\bar{b}\cdot\nabla)\bar{a} - (\bar{a}\cdot\nabla)\bar{b}$$

proof:- curl $(\bar{a}\times\bar{b}) = \sum \bar{i}\times\frac{\partial}{\partial x}(\bar{a}\times\bar{b}) = \sum \bar{i}\times\left[\frac{\partial\bar{a}}{\partial x}\times\bar{b} + \bar{a}\times\frac{\partial\bar{b}}{\partial x}\right]$

$$= \sum\bar{i}\times\left(\frac{\partial\bar{a}}{\partial x}\times\bar{b}\right) + \sum\bar{i}\times\left(\bar{a}\times\frac{\partial\bar{b}}{\partial x}\right).$$

$[A\times(B\times C) = (A\cdot C)B - (A\cdot B)C$ and $(A\times B)\times C = (A\cdot C)B - (B\cdot C)A$.

$$= \sum\left[(\bar{i}\cdot\bar{b})\frac{\partial\bar{a}}{\partial x} - \left(\bar{i}\cdot\frac{\partial\bar{a}}{\partial x}\right)\bar{b}\right] + \sum\left[\left(\bar{i}\cdot\frac{\partial\bar{b}}{\partial x}\right)\bar{a} - (\bar{i}\cdot\bar{a})\frac{\partial\bar{b}}{\partial x}\right]$$

$$= \sum(\bar{b}\cdot\bar{i})\frac{\partial\bar{a}}{\partial x} - \sum\left(\bar{i}\cdot\frac{\partial\bar{a}}{\partial x}\right)\bar{b} + \sum\left(\bar{i}\cdot\frac{\partial\bar{b}}{\partial x}\right)\bar{a} - \left(\bar{a}\cdot\sum\bar{i}\frac{\partial}{\partial x}\right)\bar{b}.$$

$$= \sum\left(\bar{b}\cdot\bar{i}\frac{\partial}{\partial x}\right)\bar{a} - \sum\left(\bar{i}\cdot\frac{\partial\bar{a}}{\partial x}\right)\bar{b} + \sum\left(\bar{i}\cdot\frac{\partial\bar{b}}{\partial x}\right)\bar{a} - (\bar{a}\cdot\nabla)\bar{b}$$

$$= (\bar{b}\cdot\nabla)\bar{a} - (\nabla\cdot\bar{a})\bar{b} + (\nabla\cdot\bar{b})\bar{a} - (\bar{a}\cdot\nabla)\bar{b}$$

$$= (\nabla\cdot\bar{b})\bar{a} - (\nabla\cdot\bar{a})\bar{b} + (\bar{b}\cdot\nabla)\bar{a} - (\bar{a}\cdot\nabla)\bar{b}$$

$$= \bar{a}\,\text{div}\,\bar{b} - \bar{b}\,\text{div}\,\bar{a} + (\bar{b}\cdot\nabla)\bar{a} - (\bar{a}\cdot\nabla)\bar{b}$$

8). If f and g are two scalar point functions. Prove that

$$\text{div}(f\nabla g) = f\nabla^2 g + \nabla f\cdot\nabla g.$$

Sol:- L.H.S. $\text{div}(f\nabla g) = \nabla\cdot(f\nabla g)$.

$$\nabla g = \bar{i}\frac{\partial g}{\partial x} + \bar{j}\frac{\partial g}{\partial y} + \bar{k}\frac{\partial g}{\partial z}$$

$$f\nabla g = \bar{i}f\frac{\partial g}{\partial x} + \bar{j}f\frac{\partial g}{\partial y} + \bar{k}f\frac{\partial g}{\partial z} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k} \text{ (say)}.$$

$$\nabla\cdot(f\nabla g) = \frac{\partial}{\partial x}\left(f\frac{\partial g}{\partial x}\right) + \frac{\partial}{\partial y}\left(f\frac{\partial g}{\partial y}\right) + \frac{\partial}{\partial z}\left(f\frac{\partial g}{\partial z}\right).$$

$$= f\frac{\partial^2 g}{\partial x^2} + \frac{\partial f}{\partial x}\cdot\frac{\partial g}{\partial x} + f\frac{\partial^2 g}{\partial y^2} + \frac{\partial f}{\partial y}\cdot\frac{\partial g}{\partial y} + f\frac{\partial^2 g}{\partial z^2} + \frac{\partial f}{\partial z}\cdot\frac{\partial g}{\partial z}$$

$$= f\left(\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} + \frac{\partial^2 g}{\partial z^2}\right) + \left(\frac{\partial f}{\partial x}\cdot\frac{\partial g}{\partial x} + \frac{\partial f}{\partial y}\cdot\frac{\partial g}{\partial y} + \frac{\partial f}{\partial z}\cdot\frac{\partial g}{\partial z}\right)$$

$$= f\nabla^2 g + \left(\bar{i}\frac{\partial f}{\partial x} + \bar{j}\frac{\partial f}{\partial y} + \bar{k}\frac{\partial f}{\partial z}\right)\cdot\left(\bar{i}\frac{\partial g}{\partial x} + \bar{j}\frac{\partial g}{\partial y} + \bar{k}\frac{\partial g}{\partial z}\right)$$

$$= f\nabla^2 g + \nabla f\cdot\nabla g$$

$$= \text{R.H.S}$$

Hence theorem proved

9. Prove that $\nabla\times(\nabla\times\bar{a}) = \nabla(\nabla\cdot\bar{a}) - \nabla^2\bar{a}$ (IV-11-i)

Sol:- L.H.S $\nabla\times(\nabla\times\bar{a})$

$$\nabla\times\bar{a} = \bar{i}\times\frac{\partial\bar{a}}{\partial x} + \bar{j}\times\frac{\partial\bar{a}}{\partial y} + \bar{k}\times\frac{\partial\bar{a}}{\partial z}$$

$$\nabla\times(\nabla\times\bar{a}) = \Sigma\,\bar{i}\times\frac{\partial}{\partial x}\left(\bar{i}\times\frac{\partial\bar{a}}{\partial x} + \bar{j}\times\frac{\partial\bar{a}}{\partial y} + \bar{k}\times\frac{\partial\bar{a}}{\partial z}\right)$$

$$= \Sigma\,\bar{i}\times\left(\bar{i}\times\frac{\partial^2\bar{a}}{\partial x^2} + \bar{j}\times\frac{\partial^2\bar{a}}{\partial x\partial y} + \bar{k}\times\frac{\partial^2\bar{a}}{\partial x\partial z}\right)$$

$$= \Sigma\left[\bar{i}\times\left(\bar{i}\times\frac{\partial^2\bar{a}}{\partial x^2}\right) + \bar{i}\times\left(\bar{j}\times\frac{\partial^2\bar{a}}{\partial x\partial y}\right) + \bar{i}\times\left(\bar{k}\times\frac{\partial^2\bar{a}}{\partial x\partial z}\right)\right]$$

$$\left[\because a\times(b\times c) = (a\cdot c)b - (a\cdot b)c\right]$$

$$= \Sigma\left[\left(\bar{i}\cdot\frac{\partial^2\bar{a}}{\partial x^2}\right)\bar{i} - (\bar{i}\cdot\bar{i})\frac{\partial^2\bar{a}}{\partial x^2} + \left(\bar{i}\cdot\frac{\partial^2\bar{a}}{\partial x\partial y}\right)\bar{j} - (\bar{i}\cdot\bar{j})\frac{\partial^2\bar{a}}{\partial x\partial y} + \left(\bar{i}\cdot\frac{\partial^2\bar{a}}{\partial x\partial z}\right)\bar{k} - (\bar{i}\cdot\bar{k})\frac{\partial^2\bar{a}}{\partial x\partial z}\right]$$

$$= \Sigma\left[\left(\bar{i}\cdot\frac{\partial^2\bar{a}}{\partial x^2}\right)\bar{i} - \frac{\partial^2\bar{a}}{\partial x^2} + \left(\bar{i}\cdot\frac{\partial^2\bar{a}}{\partial x\partial y}\right)\bar{j} + \left(\bar{i}\cdot\frac{\partial^2\bar{a}}{\partial x\partial z}\right)\bar{k}\right]$$

$$= \Sigma\left[\bar{i}\frac{\partial}{\partial x}\left(\bar{i}\cdot\frac{\partial\bar{a}}{\partial x}\right) + \bar{j}\frac{\partial}{\partial y}\left(\bar{i}\cdot\frac{\partial\bar{a}}{\partial x}\right) + \bar{k}\frac{\partial}{\partial z}\left(\bar{i}\cdot\frac{\partial\bar{a}}{\partial x}\right) - \frac{\partial^2\bar{a}}{\partial x^2}\right]$$

$$= \Sigma\left[\nabla\left(\bar{i}\cdot\frac{\partial\bar{a}}{\partial x}\right) - \frac{\partial^2\bar{a}}{\partial x^2}\right]$$

$$= \nabla\Sigma\,\bar{i}\cdot\frac{\partial\bar{a}}{\partial x} - \Sigma\frac{\partial^2\bar{a}}{\partial x^2}$$

$$= \nabla(\nabla\cdot\bar{a}) - \left(\frac{\partial^2\bar{a}}{\partial x^2} + \frac{\partial^2\bar{a}}{\partial y^2} + \frac{\partial^2\bar{a}}{\partial z^2}\right)$$

$$= \nabla(\nabla\cdot\bar{a}) - \nabla^2\bar{a}$$

$$\therefore \nabla\times(\nabla\times\bar{a}) = \nabla(\nabla\cdot\bar{a}) - \nabla^2\bar{a}$$

$$\text{curl}(\text{curl}\,\bar{a}) = \text{grad}(\text{div}\,\bar{a}) - \nabla^2\bar{a}$$

i) Prove that $\nabla f\times\nabla g$ is solenoidal.

Sol:- $\nabla f\times\nabla g$ is solenoidal i.e., $\text{div}(\nabla f\times\nabla g) = \bar{0}$

Now use identity

$$\text{div}(\bar{a}\times\bar{b}) = \bar{b}\cdot\text{curl}\,\bar{a} - \bar{a}\cdot\text{curl}\,\bar{b}$$

$$\text{div}(\nabla f\times\nabla g) = \nabla g\cdot\text{curl}\,\nabla f - \nabla f\cdot\text{curl}(\nabla g)$$

$$= \nabla g\cdot\text{curl}(\text{grad}\,f) - \nabla f\cdot\text{curl}(\text{grad}\,g)$$

$$= \nabla g\cdot(\bar{0}) - \nabla f\cdot(\bar{0}) = \bar{0}$$

$\therefore$ $\nabla f\times\nabla g$ is solenoidal.

Note:- If $\bar{a}$ & $\bar{b}$ are irrotational Then $\bar{a}\times\bar{b}$ is solenoidal

2) Find $(A \times \nabla)\phi$, if $A = yz^2\bar{i} - 3xz^2\bar{j} + 2xyz\bar{k}$ and $\phi = xyz$. (IV - II, ii)

Sol:- we have $(A\times\nabla)\phi = (\bar{A}\times\bar{i})\frac{\partial\phi}{\partial x} + (\bar{A}\times\bar{j})\frac{\partial\phi}{\partial y} + (\bar{A}\times\bar{k})\frac{\partial\phi}{\partial z}$

$\frac{\partial\phi}{\partial x} = \frac{\partial(xyz)}{\partial x} = yz, \quad \frac{\partial\phi}{\partial y} = xz, \quad \frac{\partial\phi}{\partial z} = xy$

$(\bar{A}\times\bar{i}) = (yz^2\bar{i} - 3xz^2\bar{j} + 2xyz\bar{k})\times\bar{i} = 3xz^2\bar{k} + 2xyz\bar{j}$

$(\bar{A}\times\bar{j}) = (yz^2\bar{i} - 3xz^2\bar{j} + 2xyz\bar{k})\times\bar{j} = yz^2\bar{k} - 2xyz\bar{i}$

$(\bar{A}\times\bar{k}) = (yz^2\bar{i} - 3xz^2\bar{j} + 2xyz\bar{k})\times\bar{k} = -yz^2\bar{j} - 3xz^2\bar{i}$.

$\therefore (A\times\nabla)\phi = (3xz^2\bar{k} + 2xyz\bar{j})yz + (yz^2\bar{k} - 2xyz\bar{i})xz + (-yz^2\bar{j} - 3xz^2\bar{i})(xy)$

$= 3xyz^3\bar{k} + 2xy^2z^2\bar{j} + xyz^3\bar{k} - 2x^2yz^2\bar{i} - xy^2z^2\bar{j} - 3x^2yz^2\bar{i}$

$(A\times\nabla)\phi = -5x^2yz^2\bar{i} + xy^2z^2\bar{j} + 4xyz^3\bar{k}$

3) Evaluate $\nabla\cdot\left[r\nabla\left(\frac{1}{r^3}\right)\right]$ where $r = \sqrt{x^2+y^2+z^2}$.

Sol:- Given $r = \sqrt{x^2+y^2+z^2}$, $r^2 = x^2+y^2+z^2$, $\frac{\partial r}{\partial x} = \frac{x}{r}$, $\frac{\partial r}{\partial y} = \frac{y}{r}$, $\frac{\partial r}{\partial z} = \frac{z}{r}$.

$\nabla\left(\frac{1}{r^3}\right) = \text{grad}(r^{-3}) = \Sigma\bar{i}\frac{\partial}{\partial x}(r^{-3}) = \Sigma\bar{i}(-3r^{-4})\frac{\partial r}{\partial x} = \Sigma\bar{i}(-3r^{-4})\frac{x}{r}$

$= -3r^{-5}(x\bar{i} + y\bar{j} + z\bar{k})$.

Now $r\nabla\left(\frac{1}{r^3}\right) = -3r^{-4}(x\bar{i} + y\bar{j} + z\bar{k}) = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$

$\nabla\cdot\left[r\nabla\left(\frac{1}{r^3}\right)\right] = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} = \Sigma\frac{\partial f_1}{\partial x}$

Now $f_1 = -3r^{-4}x$

$\therefore \frac{\partial f_1}{\partial x} = -3r^{-4}\cdot 1 + 12r^{-5}x\cdot\frac{\partial r}{\partial x} = -3r^{-4} + 12r^{-5}\cdot x\frac{x}{r} = -3r^{-4} + 12r^{-6}x^2$

Thus $\Sigma\frac{\partial f_1}{\partial x} = 3(-3r^{-4}) + 12r^{-6}\Sigma x^2 = -9r^{-4} + 12r^{-6}r^2 = -9r^{-4} + 12r^{-4} = 3r^{-4}$.

Hence $\nabla\cdot\left[r\nabla\left(\frac{1}{r^3}\right)\right] = \frac{3}{r^4}$

4). Find $(A\cdot\nabla)\phi$ at $(1,-1,1)$ if $A = 3xyz^2\bar{i} + 2xy^3\bar{j} - x^2yz\bar{k}$ and $\phi = 3x^2 - yz$.

Sol:- Given $A = 3xyz^2\bar{i} + 2xy^3\bar{j} - x^2yz\bar{k}$ and $\phi = 3x^2 - yz$.

$(A\cdot\nabla)\phi = (A\cdot\bar{i})\frac{\partial\phi}{\partial x} + (A\cdot\bar{j})\frac{\partial\phi}{\partial y} + (A\cdot\bar{k})\frac{\partial\phi}{\partial z}$.

$\frac{\partial\phi}{\partial x} = 6x, \quad \frac{\partial\phi}{\partial y} = -z, \quad \frac{\partial\phi}{\partial z} = -y$.

$(A\cdot\bar{i}) = (3xyz^2\bar{i} + 2xy^3\bar{j} - x^2yz\bar{k})\cdot\bar{i} = 3xyz^2$

$(A\cdot\bar{j}) = (3xyz^2\bar{i} + 2xy^3\bar{j} - x^2yz\bar{k})\cdot\bar{j} = 2xy^3$

$(A\cdot\bar{k}) = (3xyz^2\bar{i} + 2xy^3\bar{j} - x^2yz\bar{k})\cdot\bar{k} = -x^2yz$.

$\therefore (A\cdot\nabla)\phi = (3xyz^2)(6x) + (2xy^3)(-z) + (-x^2yz)(-y)$.

$(A\cdot\nabla)\phi$ at $(1,-1,1) = 18(1)(-1)(1) - 2(1)(-1)(1) + (1)(1)(1)$

$= -18 + 2 + 1 = -15$

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