UNIT- IV

BETA AND GAMMA FUNCTIONS

Beta function: The definite integral $\int_0^1 x^{m-1}(1-x)^{n-1}dx$ is called beta function and it is denoted by $\beta(m,n)$ and read as $\beta m,n$. The above integral converges $m>0$, $n>0$

Properties:

1) $\beta(m,n) = \beta(n,m)$

By def^n $\Rightarrow$ $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$

put $1-x = y \Rightarrow x = 1-y \Rightarrow dx = -dy$

when $x=0$, $y=1$

$x=1$, $y=0$

$$\beta(m,n) = \int_1^0 (1-y)^{m-1}(y)^{n-1}(-dy)$$

$$= \int_0^1 (1-y)^{m-1} y^{n-1} (dy)$$

$$= \beta(n,m)$$

2) $\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$

By def^n $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$.

put $x = \sin^2\theta$.

$dx = 2\sin\theta\cos\theta \, d\theta$

when $x=0$, $\theta=0$.

$x=1$, $\theta = \frac{\pi}{2}$.

$$\beta(m,n) = \int_0^{\pi/2} (\sin^2\theta)^{m-1} (1-\sin^2\theta)^{n-1}\, 2\sin\theta\cos\theta\, d\theta$$

$$= 2\int_0^{\pi/2} (\sin\theta)^{2m-2} (\cancel{1-}\cos\theta)^{2n-2} \sin\theta\cos\theta\, d\theta$$

$$\beta(m,n) = 2\int_0^{\pi/2} (\sin\theta)^{2m-1} (\cos\theta)^{2n-1}\, d\theta.$$

3) Other forms of Beta functions:

I^{st} form:

S.T $\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx.$

By def^n $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx.$

put $x = 1/1+y$

$dx = -\frac{1}{(1+y)^2}\,dy$

when $x=0 \Rightarrow y=\infty$

$x=1 \Rightarrow y=0$

$$\beta(m,n) = \int_\infty^0 \left(\frac{1}{1+y}\right)^{m-1}\left(1-\frac{1}{1+y}\right)^{n-1}\left(-\frac{1}{(1+y)^2}\right)dy$$

$$= \int_\infty^0 \left(\frac{1}{1+y}\right)^{m-1}\left(\frac{y}{1+y}\right)^{n-1}\left(\frac{-1}{(1+y)^2}\right)dy$$

$$= \int_\infty^0 \frac{-y^{n-1}}{(1+y)^{m\cancel{-1}+n\cancel{-1}\cancel{+2}}}\,dy$$

$$= \int_0^\infty \frac{y^{n-1}}{(1+y)^{n+m}}\,dy$$

II^{nd} form:

S.T $\beta(m$

we know t

$\Rightarrow$

put $x = \frac{1}{y}.$

$\therefore \int_1^\infty \frac{x^{m-}}{(1+x)^{n}}$

$\int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}$

Substitute (

$\therefore \beta(m,n) =$

$\underline{II^{nd}\text{ form}}$:

S.T $\beta(m,n) = \int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}}\, dx.$

we know that $\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(x+1)^{m+n}}\, dx.$

$$\Rightarrow \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}\, dx + \underbrace{\int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\, dx} \rightarrow (1)$$

put $x = \frac{1}{y} \Rightarrow dx = -\frac{1}{y^2}\, dy.$ when $x = 0 \Rightarrow y = \infty$

$x = 1 \Rightarrow y = 1$

$x = \infty \Rightarrow y = 0.$

$$\therefore \int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\, dx = \int_1^0 \frac{\left(\frac{1}{y}\right)^{m-1}}{\left(1+\frac{1}{y}\right)^{m+n}} \left(-\frac{1}{y^2}\right) dy$$

$$= \int_1^0 \frac{y^{m+n}}{y^{m-1}(1+y)^{m+n}} \left(-\frac{1}{y^2}\right) dy.$$

$$= \int_1^0 \frac{-y^{m+n}}{(1+y)^{m+n}\, y^{m+1}}\, dy.$$

$$\int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}\, dx \Leftarrow = \int_0^1 \frac{y^{n-1}}{(1+y)^{m+n}}\, dy \rightarrow (2)$$

Substitute (2) in place of (1)

$$\therefore \beta(m,n) = \int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}}\, dx.$$

IIIrd form: S.T $\beta(m,n) = a^m b^m \int_0^\infty \frac{x^{m-1}}{(ax+b)^{m+n}} dx$.

Now $a^m b^n \int_0^\infty \frac{x^{m-1}}{(ax+b)^{m+n}} dx$.

$$= a^m b^n \int_0^\infty \frac{x^{m-1}}{b^{m+n}\left(1+\frac{ax}{b}\right)^{m+n}} dx.$$

put $\frac{ax}{b} = y \Rightarrow x = \frac{b}{a} y$

$$dx = \frac{b}{a} dy$$

when $x = 0 \Rightarrow y = 0$

$x = \infty \Rightarrow y = \infty$

$$\therefore \beta(m,n) = a^m b^n \int_0^\infty \frac{\left(\frac{b}{a}y\right)^{m-1}}{b^{m+n}(1+y)^{m+n}} \left(\frac{b}{a} dy\right)$$

$$= a^m b^m \int_0^\infty \frac{b^{m-1}\, y^{m-1}}{b^{m+n}\, a^{m-1}(1+y)^{m+n}} \left(\frac{b}{a}\right) dy$$

$$= a^m b^m \int_0^\infty \frac{b^m\, y^{m-1}}{b^{m+n}\, a^m\,(1+y)^{m+n}} dy$$

$$= a^m b^m \int_0^\infty \frac{b^m\, y^{m-1}}{b^m b^n a^m (1+y)^{m+n}} dy.$$

$$= \frac{\cancel{a^m}\,\cancel{b^n}\,\cancel{b^m}}{\cancel{b^m}\,\cancel{b^n}\,\cancel{a^m}} \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}} dy$$

$$\beta(m,n) = \beta(m,n)$$

$\underline{IV^{th}\text{ form}}$: S.T $\displaystyle\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(x+a)^{m+n}}\,dx = \frac{\beta(m,n)}{a^n(1+a)^m}$

By def^n $\displaystyle\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$

put $\displaystyle x = \frac{(1+a)t}{(t+a)} = \frac{t+at}{t+a} \Rightarrow dx = \frac{(t+a)(1+a) - (t+at)(1)}{(t+a)^2}$

$\displaystyle \cancel{dx = \frac{(t+a)(t) - (t+at)(1)}{(t+a)^2}\,da}$

$\displaystyle = \frac{t+at+a+a^2 - t - at}{(t+a)^2}$

$\displaystyle dx = \frac{a+a^2}{(t+a)^2}\,dt$

when $x = 0 \Rightarrow t = 0$
$x = 1 \Rightarrow t = 1$

$$\Rightarrow \int_0^1 \left(\frac{(1+a)t}{(t+a)}\right)^{m-1}\left(1 - \left(\frac{(1+a)t}{(t+a)}\right)\right)^{n-1}\left(\frac{a+a^2}{(t+a)^2}\right)dt = \beta(m,n)$$

$$\Rightarrow \int_0^1 \frac{(1+a)^{m-1}\,t^{m-1}}{(t+a)^{m-1}}\left(\frac{a^{n-1}(1-t)^{n-1}}{(t+a)^{n-1}}\right)\frac{a+a^2}{(t+a)^2}\,dt = \beta(m,n).$$

$$= \int_0^1 \frac{(1+a)^{m-1}a^{n-1}t^{m-1}(1-t)^{n-1}\,a(1+a)}{(t+a)^{m-\cancel{1}+n-\cancel{1}+\cancel{2}}}\,dt = \beta(m,n).$$

$$\Rightarrow \int_0^1 \frac{(1+a)^{m-1}(1+a)\,a^{n-1}a\,t^{m-1}(1-t)^{n-1}}{(t+a)^{m+n}}\,dt = \beta(m,n)$$

$$\Rightarrow (1+a)^m a^n \int_0^1 \frac{(t)^{m-1}(1-t)^{n-1}}{(t+a)^{m+n}}\,dt = \beta(m,n)$$

$$\Rightarrow \int_0^1 \frac{t^{m-1}(1-t)^{n-1}}{(t+a)^{m+n}}\,dt = \frac{\beta(m,n)}{(1+a)^m a^n}$$

$$\Rightarrow \int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(x+a)^{m+n}}\,dx = \frac{\beta(m,n)}{(1+a)^m a^n}$$

V^{th} form: S.T $\int_b^a (x-b)^{m-1}(a-x)^{n-1}dx = (a-b)^{m+n-1}\beta(m,n)$

By $def^n \Rightarrow \beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$

put $x = \frac{t-b}{a-b}$

$$dx = \frac{a-b(1)-(t-b)(0)}{(a-b)^2}dt$$

$$dx = \frac{\cancel{a-b}}{(a-b)^{\cancel{2}}}dt = \frac{dt}{a-b}$$ $a-\cancel{b}-t+\cancel{b}$

$\Rightarrow$ when $x=0 \Rightarrow t=b$

$x=1 \Rightarrow t=a$

$$\therefore \beta(m,n) = \int_b^a \left(\frac{t-b}{a-b}\right)^{m-1}\left(1-\frac{t-b}{a-b}\right)^{n-1}\left(\frac{dt}{a-b}\right)$$

$$= \int_b^a \frac{(t-b)^{m-1}}{(a-b)^{m-1}}\frac{(a-t)^{n-1}}{(a-b)^{n-1}}\frac{dt}{(a-b)^1}$$

$$= \int_b^a \frac{(t-b)^{m-1}(a-t)^{n-1}}{(a-b)^{m\cancel{-1}+n\cancel{-1}+\cancel{1}-1}}dt$$

$$\Rightarrow \int_b^a \frac{(t-b)^{m-1}(a-t)^{n-1}}{(a-b)^{m+n-1}}dt = \beta(m,n)$$

$$\Rightarrow \int_b^a \frac{(x-b)^{m-1}(a-x)^{n-1}}{(a+b)^{m+n-1}}dx = \beta(m,n)$$

$$\Rightarrow \int_b^a (x-b)^{m-1}(a-x)^{n-1}dx = (a+b)^{m+n-1}\beta(m,n)$$

S.T $\int_0^{\pi/2} \sin^m\theta \cos^n\theta \, d\theta = \frac{1}{2}\beta\left(\frac{m+1}{2}, \frac{n+1}{2}\right)$

we know that $2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta = \beta(m,n)$

let $2m-1 = P$, $2n-1 = Q$

$2m = P+1$ $\quad 2n = Q+1$

$m = \frac{P+1}{2}$ $\quad n = \frac{Q+1}{2}$.

$\therefore \int_0^{\pi/2} \sin^P\theta \cos^Q\theta \, d\theta = \frac{1}{2}\beta\left(\frac{P+1}{2}, \frac{Q+1}{2}\right)$

$\therefore \int_0^{\pi/2} \sin^m\theta \cos^n\theta \, d\theta = \frac{1}{2}\beta\left(\frac{m+1}{2}, \frac{n+1}{2}\right)$

8) Express the following interms of β func^{n}s.

i) $\int_0^1 \frac{x}{\sqrt{1-x^2}} \, dx$.

put $x^2 = t$

$2x\,dx = dt$

$\Rightarrow x = 0 \rightarrow t = 0$

$x = 1 \rightarrow t = 1$

$\Rightarrow \int_0^1 \frac{\sqrt{t}}{\sqrt{1-t}} \left(\frac{dt}{2\sqrt{t}}\right)$

$\Rightarrow \frac{1}{2}\int_0^1 (1-t)^{-1/2} \, dt$

$\Rightarrow \frac{1}{2}\int_0^1 t^{1-1} (1-t)^{1/2-1} \, dt$

$\Rightarrow \frac{1}{2}\beta(1, +1/2)$

ii) $\int_0^3 \frac{dx}{\sqrt{9-x^2}}$

$x^2 = 9t$

$2x\,dx = 9\,dt$

$dx = \frac{9\,dt}{2x}$

$\Rightarrow x = 0 \rightarrow t = 0$

$x = 3 \rightarrow t = 1$

$\Rightarrow \int_0^1 \frac{1}{\sqrt{9-9t}} \left(\frac{9\,dt}{2x}\right)$

$\Rightarrow \int_0^1 \frac{1}{\cancel{3}(1-t)^{1/2}} \frac{\cancel{9}^{\,3}\,dt}{2\cancel{x}\,\cancel{3}\sqrt{t}}$

$\Rightarrow \frac{\cancel{3}1}{2}\int_0^1 t^{-1/2}(1-t)^{-1/2} \, dt$

$\Rightarrow \frac{1}{2}\int_0^1 t^{1/2-1}(1-t)^{1/2-1} \, dt$

$\therefore \frac{1}{2}\beta(1/2, 1/2)$

Q) Evaluate $\int_0^1 \frac{x^2}{\sqrt{1-x^5}}\,dx$ in terms of beta func^n.

put $x^5 = y$

$5x^4\,dx = dy \Rightarrow x^2\,dx = \frac{dy}{5x^2} = \frac{dy}{5y^{2/5}}$

when $x = 0 \Rightarrow y = 0$

$x = 1 \Rightarrow y = 1$

$\therefore \int_0^1 \frac{1}{\sqrt{1-y}} \frac{dy}{5y^{2/5}}$

$\Rightarrow \frac{1}{5}\int_0^1 y^{-2/5}(1-y)^{-1/2}\,dy$

$\Rightarrow \frac{1}{5}\int_0^1 y^{3/5-1}(1-y)^{1/2-1}\,dy$

$\Rightarrow \frac{1}{5}\beta\left(\frac{3}{5}, \frac{1}{2}\right).$

Q) i) $\int_0^\infty \frac{x^{m-1} - x^{n-1}}{(1+x)^{m+n}}\,dx = 0$

W.K.T $\Rightarrow \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx = \beta(m,n)$

$\therefore \int_0^\infty \frac{x^{m-1} - x^{n-1}}{(1+x)^{m+n}}\,dx = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx - \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$

$= \beta(m,n) - \beta(n,m)$

$= 0.$

ii) $\int_0^\infty \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}}\,dx = 2\beta(m,n)$

$\Rightarrow \int_0^\infty \frac{x}{(1+}$

$= \beta(m$

$= \beta(m$

$\Rightarrow 2$

GAMMA FUN

The d

gamma func

Properties:

1) $\Gamma 1 = 1$

2) $\Gamma 1/2 = \sqrt{}$

3) $\Gamma n = (n$

4) If n is

$\Rightarrow \Gamma 9/2 =$

$$\Rightarrow \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}}\,dx + \int_0^{\infty} \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$$

$$= \beta(m,n) + \beta(n,m) \qquad [\beta(m,n) = \beta(n,m)]$$

$$= \beta(m,n) + \beta(m,n)$$

$$\Rightarrow 2\beta(m,n).$$

GAMMA FUNCTION:

The definite integral $\int_0^{\infty} e^{-x} x^{n-1}\,dx$ is called the gamma funcⁿ and is denoted by Γn and read as gamma n.

Properties:

1) $\Gamma 1 = 1$
2) $\Gamma 1/2 = \sqrt{\pi}$
3) $\Gamma n = (n-1)\Gamma (n-1)$, where n is greater than 1.
4) If n is a non-negative integer $\Gamma (n+1) = n!$

$$\Rightarrow \Gamma 9/2 = (9/2 - 1)\,\Gamma (9/2 - 1)$$

$$\Rightarrow 7/2\,\Gamma 7/2 \Rightarrow \frac{7}{2}\left(\frac{7}{2}-1\right)\Gamma\left(\frac{7}{2}-1\right)$$

$$\Rightarrow \frac{7}{2}\cdot\frac{5}{2}\,\Gamma\frac{5}{2}$$

$$\Rightarrow \frac{7}{2}\cdot\frac{5}{2}\cdot\left(\frac{5}{2}-1\right)\Gamma\left(\frac{5}{2}-1\right)$$

$$\Rightarrow \frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\,\Gamma\frac{3}{2}$$

$$\Rightarrow \frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\left(\frac{3}{2}-1\right)\Gamma\left(\frac{3}{2}-1\right)$$

$$\Rightarrow \frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\,\Gamma\frac{1}{2}$$

$$\Rightarrow \frac{105}{16}\sqrt{\pi}$$

* Relaⁿ b/w Beta and Gamma funcⁿ.

$$\beta(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma m+n}$$

where m and n are greater than 0.

<u>Note:</u> $\Gamma n \, \Gamma 1-n = \dfrac{\pi}{\sin n\pi}$

Q) S.T $\Gamma 1/2 = \sqrt{\pi}$

W.K.T $\quad \beta(m,n) = \dfrac{\Gamma m \, \Gamma n}{\Gamma m+n} \rightarrow (1)$

put $m = 1/2$, $n = 1/2$ in eqⁿ(1)

$$\beta(1/2, 1/2) = \frac{\Gamma 1/2 \, \Gamma 1/2}{\Gamma 1/2 + 1/2}$$

$$= \frac{(\Gamma 1/2)^2}{\Gamma 1}$$

$$\beta(1/2, 1/2) = (\Gamma 1/2)^2 \rightarrow (2)$$

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx.$$

$$= \int_0^1 x^{-1/2}(1-x)^{-1/2}\,dx.$$

put $x = \sin^2\theta \qquad \Rightarrow x = 0 \rightarrow \theta = 0$

$dx = 2\sin\theta\cos\theta\,d\theta \qquad x = 1 \rightarrow \theta = \pi/2$

$$\beta(1/2, 1/2) = \int_0^{\pi/2} (\sin^2\theta)^{-1/2}(\cos^2\theta)^{-1/2}\, 2\sin\theta\cos\theta\,d\theta$$

$$= \int_0^{\pi/2} \frac{1}{\cancel{\sin\theta}} \frac{1}{\cancel{\cos\theta}}\, 2\cancel{\sin\theta}\cancel{\cos\theta}\,d\theta$$

$= 2\int_0^{\pi/2} d\theta$

$= [\theta]_0^{\pi/2}$

$= \frac{\pi}{2} \times 2$

$= \pi \rightarrow$

$\therefore (\Gamma 1/2)^2 = \pi$

$\Rightarrow \Gamma 1/2 = \sqrt{\pi}$

Q) S.T $\int_0^\infty e$

put x^2

$2x$

$\Rightarrow \int_0^\infty e^{-t}$

Q) P.T $\int_{-\infty}^0$

$\Rightarrow \int_\infty^0 e^-$

$\Rightarrow \int_0^\infty e^-$

$$= 2\int_0^{\pi/2} d\theta$$

$$= [\theta]_0^{\pi/2}$$

$$= \frac{\pi}{2} \times 2$$

$$= \pi \rightarrow (3)$$

$$\therefore \left(\sqrt{\tfrac{1}{2}}\right)^2 = \pi$$

$$\Rightarrow \Gamma\tfrac{1}{2} = \sqrt{\pi}$$

Q) S.T $\int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}.$

put $x^2 = t$ $\Rightarrow x = 0 \rightarrow t = 0$

$2x\,dx = dt$ $x = \infty \rightarrow t = \infty.$

$dx = \frac{dt}{2x} = \frac{dt}{2\sqrt{t}}$

$$\Rightarrow \int_0^{\infty} e^{-t} \frac{dt}{2\sqrt{t}} = \frac{1}{2}\int_0^{\infty} e^{-t} (t)^{-1/2} dt$$

$$= \frac{1}{2}\int_0^{\infty} e^{-t} (t)^{1/2 - 1} dt$$

$$= \frac{1}{2}\Gamma\tfrac{1}{2} = \frac{\sqrt{\pi}}{2}.$$

Q) P.T $\int_{-\infty}^{0} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}.$

$$\Rightarrow \int_{\infty}^{0} e^{-x^2}(-dx) = \int_0^{\infty} e^{-x^2} dx$$

put $x^2 = t$

$2x\,dx = dt$

$dx = \frac{dt}{2\sqrt{t}}$

$$\Rightarrow \int_0^{\infty} e^{-t} \frac{dt}{2\sqrt{t}} \Rightarrow \frac{1}{2}\int_0^{\infty} e^{-t}(t)^{1/2 - 1} dt$$

$$\Rightarrow \frac{1}{2} \times \Gamma\tfrac{1}{2} = \frac{\sqrt{\pi}}{2}.$$

Q) $\int_{-\infty}^{\infty} e^{-x^2} dx$

$\Rightarrow \int_{-\infty}^{0} e^{-x^2} dx + \int_{0}^{+\infty} e^{-x^2} dx$

$\Rightarrow \int_{\infty}^{0} e^{-x^2} (-dx) + \int_{0}^{\infty} e^{-x^2} dx$

$\Rightarrow 2\int_{0}^{\infty} e^{-x^2} (dx) = 2\int_{0}^{\infty} e^{-t} \frac{1}{2\sqrt{t}} dt$

$= 2 \times \frac{1}{2} \Gamma(1/2) = \sqrt{\pi}$

Q) Find the value of $\Gamma(11/2)$, $\Gamma(-1/2)$, $\Gamma(5/2)$, $\Gamma(-7/2)$, $\Gamma(10)$

i) $\Gamma(11/2) = \left(\frac{11}{2}-1\right)\Gamma\left(\frac{11}{2}-1\right) = \frac{9}{2}\cdot\left(\frac{9}{2}-1\right)\Gamma\left(\frac{9}{2}-1\right) = \frac{9}{2}\cdot\frac{7}{2}\cdot\left(\frac{7}{2}-1\right)\Gamma\left(\frac{7}{2}-1\right)$

$\Rightarrow \frac{9}{2}\cdot\frac{7}{2}\cdot\frac{5}{2}\left(\frac{5}{2}-(1)\right)\Gamma\left(\frac{5}{2}-1\right)$

$\Rightarrow \frac{9}{2}\cdot\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma(1/2)$

$\Rightarrow \frac{945}{32}\sqrt{\pi}$

ii) $\Gamma(-1/2) \Rightarrow \Gamma(n+1) = n\Gamma(n)$

$\Gamma(n) = \frac{\Gamma(n+1)}{n}$

$\Rightarrow \Gamma(-1/2) = \frac{\Gamma(-1/2+1)}{-1/2} = \frac{\Gamma(1/2)}{-1/2} = \frac{\sqrt{\pi}}{-1/2} = -2\sqrt{\pi}$

iii) $\Gamma(5/2) \Rightarrow \left(\frac{5}{2}-1\right)\Gamma\left(\frac{5}{2}-1\right) \Rightarrow \frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right) \Rightarrow \frac{3}{4}\sqrt{\pi}$

iv) $\Gamma\left(-\frac{7}{2}\right)$ $\quad \Gamma n = \frac{\Gamma(n+1)}{n}$

$$\Rightarrow \frac{\Gamma\left(-\frac{7}{2}+1\right)}{-\frac{7}{2}} \Rightarrow \frac{\Gamma\left(-5/2\right)}{-\frac{7}{2}} \Rightarrow \frac{\Gamma\left(-\frac{5}{2}+1\right)}{-\frac{5}{2}\cdot-\frac{7}{2}} = \frac{\Gamma\left(\frac{-3}{2}\right)}{\frac{35}{4}}$$

$$\Rightarrow \frac{\Gamma\left(-\frac{3}{2}+1\right)}{-\frac{3}{2}\cdot\frac{35}{4}} \Rightarrow \frac{\Gamma\left(-\frac{1}{2}\right)}{-\frac{3}{2}\cdot\frac{35}{4}} \Rightarrow \frac{\Gamma\left(-\frac{1}{2}+1\right)}{-\frac{3}{2}\cdot-\frac{1}{2}\cdot\frac{35}{4}} = \frac{\Gamma\left(\frac{1}{2}\right)}{\frac{105}{16}}$$

$$= \frac{16}{105}\sqrt{\pi}$$

v) $\Gamma 10 = 9\Gamma 9 = 9\cdot 8\,\Gamma 7 \Rightarrow 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1\,\Gamma 1$

$$\Rightarrow 362880$$

Relation b/w Beta and Gamma:

$$\beta(m,n) = \frac{\Gamma m\,\Gamma n}{\Gamma(m+n)}$$

By def $\Gamma m = \int_0^\infty e^{-x} x^{m-1}\,dx \rightarrow (1)$

$$x = yt \Rightarrow dx = y\,dt$$

when $x = 0 \rightarrow t = 0$

$x = \infty \rightarrow t = \infty$

$$\Rightarrow \Gamma m = \int_0^\infty e^{-yt} (yt)^{m-1} (y\,dt)$$

$$= \int_0^\infty e^{-yt} y^m t^{m-1}\,dt$$

$$\Rightarrow y^m \int_0^\infty e^{-yt} t^{m-1}\,dt$$

$$\frac{\Gamma m}{y^m} = \int_0^\infty e^{-yt} t^{m-1}\,dt \rightarrow (2)$$

Multiplying eq^n (2) with $e^{-y}\cdot y^{m+n-1}$ on both sides and integrating w.r.t y from $0 \to \infty$

$$\frac{\Gamma m}{y^m} \times e^{-y} \cdot y^{m+n-1} \cancel{dy} = \int_0^\infty e^{-yx} x^{m-1} \times e^{-y} \times y^{m+n-1} \cancel{dy}\, dx$$

$$\Rightarrow \quad \Gamma m \, e^{-y} y^{n-1} = \int_0^\infty e^{-y(1+x)} y^{m+n-1} x^{m-1} \, dx$$

$\Rightarrow$ Integrating on both sides w.r.t ~~to~~ y from $0 \to \infty$.

$$\Rightarrow \int_0^\infty \Gamma m \, e^{-y} y^{n-1} \, dy = \int_0^\infty \left[\int_0^\infty e^{-y(1+x)} y^{m+n-1} \, dy \right] x^{m-1} \, dx$$

$$\Rightarrow \Gamma m \int_0^\infty e^{-y} y^{n-1} \, dy = \int_0^\infty \frac{\Gamma(m+n)}{(1+x)^{m+n}} x^{m-1} \, dx$$

$$\Rightarrow \Gamma m \, \Gamma n = \Gamma(m+n) \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} \, dx$$

$$\Rightarrow \frac{\Gamma m \, \Gamma n}{\Gamma(m+n)} = \beta(m,n) \quad //.$$

Q) Evaluate i) $\int_0^1 x^5 (1-x)^3 \, dx$

i) $\cancel{\beta(m,n) = \int_0^\infty e^{-x} x^{n-1} d}$

$$\beta(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} \, dx.$$

$$\Rightarrow \int_0^1 x^{6-1} (1-x)^{4-1} \, dx$$

$$\Rightarrow \beta(6,4) = \frac{\Gamma 6 \, \Gamma 4}{\Gamma 10} = \frac{5!\,3!}{9!} = \frac{1}{504}$$

ii) $\int_0^1 x^4 (1-x)^2 dx$

$\Rightarrow \int_0^1 x^{5-1}(1-x)^{3-1} dx = \beta(5,3) = \frac{\Gamma 5\,\Gamma 3}{\Gamma 8} = \frac{4!\,2!}{7!}$

$= \frac{1}{105}$

iii) $\int_0^2 x(8-x^3)^{1/3} dx$.

$x^3 = 8y \Rightarrow 3x^2 dx = 8\,dy$

$dx = \frac{8\,dy}{3(8y)^{2/3}} = \frac{\cancel{8}^{2}\,dy}{3\times\cancel{4}(y)^{2/3}}$

$\Rightarrow x = 0 \to y = 0$

$x = 2 \to y = 1$

$= \frac{2}{3} y^{-2/3} dy$

$= \frac{2}{3} y^{-2/3} dy$.

$\Rightarrow \int_0^1 2y^{1/3}(8-8y)^{1/3} \times \frac{2}{3} \times y^{-2/3} dy$

$\Rightarrow \frac{4}{3}\int_0^1 y^{-1/3} \cdot 8^{1/3}(1-y)^{1/3} dy$

$\Rightarrow \frac{8}{3}\int_0^1 y^{-1/3}(1-y)^{1/3} dy = \frac{8}{3}\int_0^1 y^{2/3-1}(1-y)^{4/3-1} dy$

$\Rightarrow \frac{8}{3}\beta\left(\frac{2}{3}, \frac{4}{3}\right)$

$\Rightarrow \frac{8}{3}\frac{\Gamma\frac{2}{3}\,\Gamma\frac{4}{3}}{\Gamma 2}$

$\Rightarrow \frac{8}{3}\frac{\Gamma\frac{2}{3}\,\Gamma\left(\frac{4}{3}-1\right)\left(\frac{4}{3}-1\right)}{1!}$

$\Rightarrow \frac{8}{3}\cdot\frac{1}{3}\frac{\Gamma\frac{2}{3}\,\Gamma\frac{1}{3}}{1!} \Rightarrow \frac{8}{9}\Gamma\frac{2}{3}\,\Gamma\left(1-\frac{2}{3}\right) = \frac{8}{9}\frac{\pi}{\sin 2\pi/3} = \frac{8}{9}\times\frac{2}{\sqrt{3}}\pi$

$\Rightarrow \frac{16\pi}{9\sqrt{3}}$

iv) $\int_0^1 x^{5/2}(1-x^2)^{3/2}\,dx$.

put $x^2 = y$ $\Rightarrow$ $x=0 \Rightarrow y=0$

$2x\,dx = dy$ $\quad x=1 \Rightarrow y=1$

$dx = \frac{dy}{2\sqrt{y}}$

$$\Rightarrow \int_0^1 (y^{1/2})^{5/2}(1-y)^{3/2}\frac{dy}{2y^{1/2}}.$$

$$\Rightarrow \int_0^1 y^{5/4}\, y^{-1/2}(1-y)^{3/2}\frac{dy}{2}$$

$$\Rightarrow \frac{1}{2}\int_0^1 y^{3/2}(1-y)^{3/2}\,dy = \frac{1}{2}\int_0^1 y^{5/2-1}(1-y)^{5/2-1}\,dy$$

$$\Rightarrow \frac{1}{2}\beta\left(\frac{5}{2},\frac{5}{2}\right) = \frac{1}{2}\frac{\Gamma\frac{5}{2}\,\Gamma\frac{5}{2}}{\Gamma 5} = \frac{1}{2}\frac{\left(\frac{5}{2}-1\right)\Gamma\left(\frac{5}{2}-1\right)\left(\frac{5}{2}-1\right)\Gamma\left(\frac{5}{2}-1\right)}{4!}$$

$$\Rightarrow \frac{1}{2}\cdot\frac{3}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\;\frac{\Gamma\frac{1}{2}\,\Gamma\frac{1}{2}}{4!}$$

$$\Rightarrow \frac{\cancel{9}^{\,1}}{4\,\cancel{32}}\frac{\pi}{4!} \Rightarrow \frac{\pi}{96}.$$

If $\int_0^\infty e^{-kx}x^{n-1}\,dx = \frac{\Gamma n}{(k)^n}$

Q) compute 1) $\int_0^\infty e^{-x}x^3\,dx$

2) $\int_0^\infty x^4 e^{-2x}\,dx$

3) $\int_0^\infty e^{-4x}x^{3/2}\,dx$.

1) $\int_0^\infty e^{-x}\, x^3\, dx.$

$\therefore = \int_0^\infty e^{-x}\, x^{4-1}\, dx$

$= \Gamma 4 = 3! = 6.$

2) $\int_0^\infty e^{-2x}\, x^4\, dx$

$= \dfrac{\Gamma 5}{(2)^5} = \dfrac{4!}{(2)^5}$

3) $\int_0^\infty e^{-4x}\, x^{3/2}\, dx.$

$= \int_0^\infty e^{-4x}\, x^{\frac{5}{2}-1}\, dx$

$= \dfrac{\Gamma \frac{5}{2}}{(4)^{5/2}} = \dfrac{\frac{3}{2}!}{(4)^{5/2}}.$

Q) S.T $\Gamma n = \int_0^1 \left(\log \frac{1}{x}\right)^{n-1} dx.$

put $x = \log\frac{1}{y} = \log y^{-1} = -\log y$

$y = e^{-x}.$ $\qquad x = 0 \Rightarrow y = 1$

$\dfrac{dy}{dx} = e^{-x}(-1)$ $\qquad x = \infty \Rightarrow y = 0.$

$dx = \dfrac{-1}{e^{-x}}\, dy$

$dx = -\dfrac{1}{y}\, dy$

By def $\Gamma n = \int_0^\infty e^{-x} x^{n-1}\, dx$

$\Gamma n = \int_1^0 y\,(-\log y)^{n-1} \left(-\frac{1}{y}\right) dy$

$= \int_1^0 \cancel{y} \left(\log \frac{1}{y}\right)^{n-1} \left(-\frac{1}{\cancel{y}}\right) dy = \int_0^1 \left(\log \frac{1}{y}\right)^{n-1} dy.$

$\therefore \Gamma n = \int_0^1 \left(\log \frac{1}{y}\right)^{n-1} dy.$

Q) S.T $\Gamma n = \frac{1}{n} \int_0^\infty e^{-x^{1/n}} dx$.

Put $x^n = t$

$$n(x)^{n-1} dx = dt. \Rightarrow x^{n-1} dx = \frac{1}{n} dt$$

$$dx = \frac{1}{n} \frac{1}{(x)^{n-1}} dt.$$

$$= \frac{1}{n} \frac{x}{x^n} dt$$

$x = 0 \to t = 0$

$x = \infty \to t = \infty$

$$= \frac{1}{n} \frac{t^{1/n}}{t} dt.$$

$$= \frac{1}{n} t^{1/n - 1} dt.$$

$$\Gamma n = \int_0^\infty e^{-x} x^{n-1} dx$$

$$= \int_0^\infty e^{-t^{1/n}} \frac{1}{n} dt$$

$$= \frac{1}{n} \int_0^\infty e^{-t^{1/n}} dt$$

$$\Gamma n = \frac{1}{n} \int_0^\infty e^{-x^{1/n}} dx$$

Q) Evaluate i) $\int_0^{\pi/2} \sin^5\theta \cos^{7/2}\theta \, d\theta$.

ii) $\int_0^{\pi/2} \sin^7\theta \, d\theta$

iii) $\int_0^{\pi/2} \cos^{11}\theta \, d\theta$.

iv) $\int_0^{\pi/2} \sqrt{\cot\theta} \, d\theta$

1) ~~β(m~~ $\int_0^{\pi/2} \sin^5\theta \cos^{7/2}\theta \, d\theta$.

$$\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta.$$

$$\therefore \Rightarrow \int_0^{\frac{\pi}{2}} \sin^{2(3)-1}\theta \cos^{2\left(\frac{9}{4}\right)-1}\theta \, d\theta = \frac{1}{2}\beta\left(3, \frac{9}{4}\right)$$

$$\Rightarrow \beta\left(3, \frac{9}{4}\right) \times \frac{1}{2} = \frac{1}{2} \frac{\Gamma 3 \, \Gamma\frac{9}{4}}{\Gamma\left(3+\frac{9}{4}\right)} = \frac{1}{\cancel{2}} \frac{\cancel{2}! \, \Gamma\frac{9}{4}}{\Gamma\frac{21}{4}}$$

$$= \frac{\cancel{\Gamma\frac{9}{4}}}{\frac{17}{4}\cdot\frac{13}{4}\cdot\frac{9}{4}\cancel{\Gamma\frac{9}{4}}} = \frac{4\times4\times4}{17\times13\times9}$$

ii) $\int_0^{\frac{\pi}{2}} \sin^7\theta \, d\theta$.

$$\Rightarrow \int_0^{\frac{\pi}{2}} \sin^{2(4)-1}\theta \cos^{2\left(\frac{1}{2}\right)-1}\theta \, d\theta$$

$$\Rightarrow \frac{1}{2}\left(\beta\left(4, \frac{1}{2}\right)\right) = \frac{1}{2} \times \frac{\Gamma 4 \, \Gamma\frac{1}{2}}{\Gamma\frac{9}{2}} = \frac{1}{\cancel{2}} \times \frac{3\times2\times\sqrt{\pi}}{\frac{7}{2}\times\frac{5}{2}\times\frac{3}{2}\times\frac{1}{\cancel{2}}\Gamma\frac{1}{2}}$$

$$= \frac{\cancel{3}\times2\times2\times2\times2\times\cancel{\sqrt{\pi}}}{7\times5\times\cancel{3}\times\cancel{\sqrt{\pi}}}$$

$$= \frac{16}{35}$$

#

iii) $\int_0^{\pi/2} \cos^{11}\theta \, d\theta$

$\Rightarrow \int_0^{\frac{\pi}{2}} \sin^{2(\frac{1}{2})-1}\theta \cos^{2(6)-1}\theta \, d\theta$

$\Rightarrow \frac{1}{2}\beta\left(\frac{1}{2}, 6\right)$

$\Rightarrow \frac{1}{2} \times \frac{\Gamma\frac{1}{2}\,\Gamma 6}{\Gamma\frac{13}{2}}$

$\Rightarrow \frac{1}{\cancel{2}} \times \frac{\cancel{\sqrt{\pi}} \times 5 \times \cancel{3} \times 4 \times 2}{\frac{11}{2} \times \frac{9}{2} \times \frac{7}{2} \times \frac{5}{2} \times \frac{\cancel{3}}{2} \times \frac{\cancel{1}}{2}\,\cancel{\Gamma\frac{1}{2}}}$

$\Rightarrow \frac{5 \times 4 \times 2 \times 8 \times 4}{11 \times 9 \times 7 \times 5}$

iv) $\int_0^{\pi/2} \sqrt{\cot\theta} \, d\theta$

$\Rightarrow \int_0^{\pi/2} \sin^{-1/2}\theta \cos^{1/2}\theta \, d\theta$

$\Rightarrow \int_0^{\pi/2} \sin^{2(\frac{1}{4})-1}\theta \cos^{2(+\frac{3}{4})-1}\theta \, d\theta$

$\Rightarrow \frac{1}{2}\beta\left(\frac{1}{4}, \frac{3}{4}\right)$

$\Rightarrow \frac{1}{2} \frac{\Gamma\frac{1}{4}\,\Gamma\frac{3}{4}}{\Gamma 1} = \frac{1}{2} \times \Gamma\frac{1}{4}\,\Gamma\left(1-\frac{1}{4}\right)$

$= \frac{1}{2} \times \frac{\pi}{\sin\frac{\pi}{4}}$

$= \frac{1}{2} \times \frac{\pi}{\frac{1}{\sqrt{2}}}$

$= \frac{\sqrt{2}\,\pi}{\cancel{2}\sqrt{2}}$

$= \frac{\pi}{\sqrt{2}}$

Q) Evaluate 1) $\int_0^\infty 3^{-4x^2} dx$

2) $\int_0^\infty a^{-bx^2} dx$

3) $\int_0^1 x^4 \left(\log \frac{1}{x}\right)^3 dx$

4) $\int_0^1 x^2 \left(\log \frac{1}{x}\right)^3 dx.$

1) $3 = e^{\log 3}$

$3^{-4x^2} = e^{-4x^2 \log 3}.$

$$\int_0^\infty 3^{-4x^2} dx = \int_0^\infty e^{-4x^2 \log 3} dx$$

Put $2x\sqrt{\log 3} = t$

$2\sqrt{\log 3}\, dx = dt$

$dx = \frac{1}{2\sqrt{\log 3}} dt.$

$x = 0 \to t = 0$

$x = \infty \to t = \infty$

$$\int_0^\infty e^{-t^2} \frac{1}{2\sqrt{\log 3}} dt = \frac{1}{2\sqrt{\log 3}} \int_0^\infty e^{-t^2} dt$$

$$= \frac{1}{2\sqrt{\log 3}} \frac{\sqrt{\pi}}{2}.$$

$$= \frac{\sqrt{\pi}}{4\sqrt{\log 3}}$$

3) $\int_0^1 x^4 \left(\log \frac{1}{x}\right)^3 dx.$

Put $y = \log \frac{1}{x}$

$y = -\log x$ $\quad\quad x = 0 \rightarrow y = \infty$

$\quad\quad x = 1 \rightarrow y = 0.$

$x = e^{-y}$

$dx = -e^{-y} dy$

$\Rightarrow \int_\infty^0 (e^{-y})^4 y^3 (-e^{-y}) dy = \int_0^\infty e^{-5y} y^{4-1} dy$

$= \frac{\Gamma 4}{(5)^4} = \frac{3!}{5^4} = \frac{6}{625}$

2) $\int_0^\infty a^{-bx^2} dx$

Let $a = e^{\log a}$

$a^{-bx^2} = e^{-bx^2 \log a}$

$\therefore \Rightarrow \int_0^\infty e^{-bx^2 \log a} dx$

Put $bx^2 \log a = t^2$ $\quad\quad x = 0 \longrightarrow t = 0$

$\quad\quad x = \infty \rightarrow t = \infty.$

$\sqrt{b\log a}\, x = t$

$\sqrt{b\log a}\, dx = dt$

$dx = \frac{1}{\sqrt{b\log a}} dt$

$\Rightarrow \int_0^\infty e^{-t^2} \left(\frac{1}{\sqrt{b\log a}}\right) dt = \frac{1}{\sqrt{b\log a}} \int_0^\infty e^{-t^2} dt$

$= \frac{\sqrt{\pi}}{2\sqrt{b\log a}}$

$\rightarrow y = 1$

$\infty \rightarrow y = 0.$

$\log\left(\frac{1}{y}\right)\Big)^{n-1} dy$

4) $\int_0^1 x^2 \left(\log \frac{1}{x}\right)^3 dx$

Put $y = \log \frac{1}{x}$.

$y = -\log x \Rightarrow x = e^{-y}$.

$dx = -e^{-y} dy$.

$\therefore$ $x = 0 \Rightarrow y = \infty$

$x = 1 \Rightarrow y = 0$.

$\Rightarrow \int_\infty^0 (e^{-y})^2 (y)^3 (-e^{-y}) dy$

$\Rightarrow \int_\infty^0 e^{-2y} y^3 (-e^{-y}) dy$

$\Rightarrow \int_0^\infty e^{-3y} y^3 dy$

$\Rightarrow \int_0^\infty e^{-3y} y^{4-1} dy$

$\Rightarrow \frac{\Gamma 4}{(3)^4} = \frac{\cancel{3}\cdot 2}{3^3 \times \cancel{3}} = \frac{2}{27}$.

$e^{-y}dy.$

5) S.T $\int_0^\infty x^4 e^{-x^2}dx = \frac{3\sqrt{\pi}}{8}$

put $x^2 = t \Rightarrow x = \sqrt{t}$

$2x\,dx = dt$ $\qquad x=0 \Rightarrow t=0$

$dx = \frac{1}{2x}dt$ $\qquad x=\infty \Rightarrow t=\infty$

$= \frac{1}{2\sqrt{t}}dt$

$\Rightarrow \int_0^\infty t^2 e^{-t}\left(\frac{1}{2\sqrt{t}}\right)dt$

$\Rightarrow \frac{1}{2}\int_0^\infty t^{3/2}e^{-t}dt$

$\Rightarrow \frac{1}{2}\int_0^\infty e^{-t}t^{\frac{5}{2}-1}dt$

$= \frac{1}{2}\Gamma\frac{5}{2} \quad \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{3}{2}\cdot\sqrt{\pi} \Rightarrow \frac{3\sqrt{\pi}}{4\cdot 2} = \frac{3}{8}\sqrt{\pi}$

$\cdot 2 = \frac{2}{27}$

$\times 3$

Q) Evaluate $4\int_0^\infty \frac{x^2}{1+x^4}dx$ using β-Γ func^n

$1-x^2 \Rightarrow x = \sin\theta$

$1+x^2 \Rightarrow x = \tan\theta$

Put $x^2 = \tan\theta \Rightarrow x = \sqrt{\tan\theta}$

~~2x dx =~~ $1+x^4 \Rightarrow 1+\tan^2\theta = \sec^2\theta.$

~~tanθ dθ~~

$\Rightarrow x^2 = \tan\theta$

$2x\,dx = \sec^2\theta\,d\theta.$ $\qquad x=0 \Rightarrow \theta=0$

$dx = \frac{1}{2\sqrt{\tan\theta}}\sec^2\theta\,d\theta$ $\qquad x=\infty \Rightarrow \theta = \frac{\pi}{2}.$

$\frac{1}{4}\int_0^\infty \frac{x^2}{1+x^4}dx = \cancel{4}\,2\int_0^{\frac{\pi}{2}} \frac{\sqrt{\tan\theta}}{1+\tan^2\theta}\,\frac{1}{2\sqrt{\tan\theta}}\sec^2\theta\,d\theta$

$\Rightarrow 2\int_0^{\frac{\pi}{2}} \sin^{1/2}\theta\cos^{-1/2}\theta\,d\theta.$

$\Rightarrow \Gamma\frac{3}{4}\Gamma\frac{1}{4} \Rightarrow \cancel{\frac{1}{4}\Gamma\frac{1}{4}\Gamma\frac{1}{4}}$ $\quad = 2\int_0^{\frac{\pi}{2}} \sin^{2(\frac{3}{4})-1}\theta\cos^{2(\frac{1}{4})-1}\theta\,d\theta$

$4 \leftarrow = \beta\left(\frac{3}{4},\frac{1}{4}\right) = \frac{\Gamma\frac{3}{4}\Gamma\frac{1}{4}}{\Gamma\frac{1}{2}} \Rightarrow$

Q) 1) $\int_0^{\pi/2} \sqrt{\cos x}\, dx \times \int_0^{\pi/2} \frac{1}{\sqrt{\cos x}}\, dx = \pi$

$$\Rightarrow \int_0^{\pi/2} \cos^{1/2} dx = \int_0^{\pi/2} \sin^{2(\frac{1}{2})-1} x \cos^{2(\frac{3}{4})-1} x\, dx$$

$$= \frac{1}{2}\beta\left(\frac{1}{2}, \frac{3}{4}\right) = \frac{1}{2} \frac{\Gamma\frac{1}{2}\, \Gamma\frac{3}{4}}{\Gamma\frac{5}{4}}$$

$$= \frac{1}{2} \times \frac{\sqrt{\pi} \times \Gamma\frac{3}{4}}{\Gamma\frac{5}{4}}$$

$$\Rightarrow \int_0^{\pi/2} \cos^{-1/2} x\, dx = \int_0^{\pi/2} \sin^{2(\frac{1}{2})-(1)} x \cos^{2(\frac{1}{4})-1} x\, dx$$

$$= \frac{1}{2}\beta\left(\frac{1}{2}, \frac{1}{4}\right)$$

$$= \frac{1}{2} \times \frac{\Gamma\frac{1}{2}\, \Gamma\frac{1}{4}}{\Gamma\frac{3}{4}}$$

$$\Rightarrow \frac{1}{2} \times \frac{\Gamma\frac{1}{2}\, \Gamma\frac{3}{4}}{\Gamma\frac{5}{4}} \times \frac{1}{2} \times \frac{\Gamma\frac{1}{2}\, \Gamma\frac{1}{4}}{\Gamma\frac{3}{4}}$$

$$\Rightarrow \frac{1}{2} \times \frac{\Gamma\frac{1}{2}\, \cancel{\Gamma\frac{3}{4}}}{\frac{1}{4}\cancel{\Gamma\frac{1}{4}}} \times \frac{1}{2} \times \frac{\Gamma\frac{1}{2}\, \cancel{\Gamma\frac{1}{4}}}{\cancel{\Gamma\frac{3}{4}}}$$

$$\Rightarrow \frac{1}{\cancel{4}} \times \pi \times \cancel{4}$$

$$= \pi$$

Q) S.T $\int_0^{\pi/2} (\sqrt{\tan\theta} + \sqrt{\sec\theta})\, d\theta$

$$= \frac{1}{2}\Gamma\frac{1}{4}\left[\Gamma\frac{3}{4} + \frac{\sqrt{\pi}}{\Gamma\frac{1}{4}}\right]$$

Q) Evaluate $\int_0^{\infty} \frac{x^4(1+x^5)}{(1+x)^{15}}\, dx$

$$\Rightarrow \int_0^{\infty} \frac{x^4}{(1+x)^{15}}\, dx + \int_0^{\infty} \frac{x^9}{(1+x)^{15}}\, dx.$$

$$= \int_0^{\infty} \frac{x^{5-1}}{(1+x)^{5+10}}\, dx + \int_0^{\infty} \frac{x^{10-1}}{(1+x)^{10+5}}\, dx.$$

$$= \beta(5,10) + \beta(10,5)$$

$$= 2\beta(5,10)$$

$$= 2\,\frac{\Gamma 5\,\Gamma 10}{\Gamma 15}.$$

$$= 2\,\frac{4!\,9!}{14!}$$

Q) P.T $\int_0^{\infty} \frac{x^8(1-x^6)}{(1+x)^{24}}\, dx = 0$

$$\int_0^{\infty} \frac{x^8}{(1+x)^{24}}\, dx + \int_0^{\infty} \frac{-x^{14}}{(1+x)^{24}}\, dx$$

$$= \int_0^{\infty} \frac{x^{9-1}}{(1+x)^{9+15}}\, dx - \int_0^{\infty} \frac{x^{15-1}}{(1+x)^{15+9}}\, dx$$

$$= \beta(9,15) - \beta(15,9)$$

$$= 0.$$

Q) S.T $\int_0^\infty \sqrt{x}\, e^{-x^3}\, dx = \frac{\sqrt{\pi}}{3}$.

$x^3 = t$ $\qquad\qquad x = 0 \Rightarrow t = 0$

$3x^2\, dx = dt$ $\qquad\qquad x = \infty \Rightarrow t = \infty$

$dx = \frac{1}{3\, t^{2/3}}\, dt$

$$\Rightarrow \int_0^\infty (t^{1/3})^{1/2} e^{-t} \left(\frac{1}{3\, t^{2/3}}\right) dt$$

$$\Rightarrow \frac{1}{3} \int_0^\infty t^{1/6}\, t^{-2/3}\, e^{-t}\, dt$$

$$\Rightarrow \frac{1}{3} \int_0^\infty t^{-\cancel{3}^{\,1}/\cancel{6}_{\,2}}\, e^{-t}\, dt$$

$$\Rightarrow \frac{1}{3} \int_0^\infty e^{-t}\, t^{\frac{1}{2}-1}\, dt$$

$$\Rightarrow \frac{1}{3} \Gamma\frac{1}{2} \Rightarrow \frac{1}{3}\sqrt{\pi}$$

$$= \frac{\sqrt{\pi}}{3}$$

MULTIPLE INTEGRALS:

The double integral of x, y over the region or is denoted by the symbol $\iint_R f(x,y)\, dR$ (or) $\iint_R f(x,y)\, dx\, dy$.

Evaluation:

Suppose that R can be described by inequalities of the form $a \le x \le b$, $y_1(x) \le y_2(x)$ so that $y = y_1(x)$ and $y = y_2(x)$ represents the boundary or then integral $\int_a^b \int_{y_1(x)}^{y_2(x)} f(x,y)\,dy\,dx$

$$= \int_a^b \left[\int_{y_1(x)}^{y_2(x)} f(x,y)\,dy \right]$$

Similarly if R can be described by the inequalities of the form $a \le y \le b$, $x_1(y) \le x_2(y)$.

$$\iint f(x,y)\,dx\,dy = \int_{y=a}^{b} \left[\int_{x=x_1(y)}^{x=x_2(y)} f(x,y)\,dx \right] dy.$$

Note: If all the four limits of integratⁿ are constants then the double integral can be evaluated in either way, we first integrate w.r.t x or y. (or)

We first integrating w.r.t & then w.r.t y.

over the

$\int_R f(x,y)\,dR$

Q) Evaluate. $\int_0^3\int_1^2 xy(1+x+y)\,dy\,dx$.

$$\int_0^3\left[\int_1^2 y(1+x+y)\,dy\right]x\,dx$$

$$\Rightarrow \int_0^3\left[\int_1^2 y+xy+y^2\,dy\right]x\,dx$$

$$\Rightarrow \int_0^3\left[\left[\frac{y^2}{2}+x\frac{y^2}{2}+\frac{y^3}{3}\right]_1^2 x\,dx\right.$$

$$\Rightarrow \int_0^3\left[\left[\frac{4}{2}+\frac{4x}{2}+\frac{8}{3}\right]-\left[\frac{1}{2}+\frac{x}{2}+\frac{1}{3}\right]\right]x\,dx$$

$$\Rightarrow \int_0^3\left[\frac{3}{2}+\frac{3x}{2}+\frac{7}{3}\right]x\,dx.$$

$$\Rightarrow \int_0^3 \frac{3x}{2}+\frac{3x^2}{2}+\frac{7x}{3}\,dx$$

$$\Rightarrow \int_0^3 \frac{9x+9x^2+14x\,dx}{6}$$

$$=\int_0^3 \frac{23x}{6}+\frac{9x^2}{6}\,dx.$$

$$\Rightarrow \left[\frac{23}{6}\frac{x^2}{2}+\frac{\cancel{9}3}{\cancel{6}2}\frac{x^3}{\cancel{3}1}\right]_0^3$$

$$=\left[\frac{23}{12}\left(\frac{9}{2}\right)+\frac{1}{2}(27)-\frac{23}{12}(0)+\frac{1}{2}(0)\right]$$

$$=\frac{23}{\cancel{12}4}\times\frac{\cancel{9}3}{2}+\frac{27}{2}\Rightarrow\frac{23\times3+27\times2}{4}=\frac{69+54}{4}=\frac{123}{4}$$

Q) Evaluate $\int_0^2 \int_0^x y\,dy\,dx$.

$$\Rightarrow \int_{x=0}^{2} \left[\int_{y=0}^{x} y\,dy \right] dx$$

$$\Rightarrow \int_0^2 \left[\frac{y^2}{2} \right]_0^x dx$$

$$\Rightarrow \int_0^2 \frac{x^2}{2}\,dx$$

$$\Rightarrow \frac{1}{2} \left[\frac{x^3}{3} \right]_0^2 = \frac{1}{2} \left[\frac{8}{3} \right] = \frac{4}{3}.$$

Q) $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{1}{1+x^2+y^2}\,dy\,dx$

$$\Rightarrow \int_0^1 \left[\int_0^{\sqrt{1+x^2}} \frac{1}{1+x^2+y^2}\,dy \right] dx$$

Put $1+x^2 = a^2$

$\sqrt{1+x^2} = a$

$\Rightarrow$

Q) $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{1}{1+x^2+y^2}\, dy\, dx$

$\Rightarrow \int_0^1 \left[\int_0^{\sqrt{1+x^2}} \frac{1}{1+x^2+y^2}\, dy \right] dx$ put $1+x^2 = a^2$, $\sqrt{1+x^2} = a$

$\Rightarrow \int_0^1 \left[\int_0^a \frac{1}{a^2+y^2}\, dy \right] dx$

$\Rightarrow \int_0^1 \left[\frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) \right]_0^a dx$

$\Rightarrow \int_0^1 \left[\frac{1}{a} \frac{\pi}{4} - 0 \right] dx.$

$\Rightarrow \frac{\pi}{4} \int_0^1 \frac{1}{\sqrt{1+x^2}} \Rightarrow \frac{\pi}{4} \left(\log\left(x + \sqrt{1+x^2}\right) \right)_0^1$

$\Rightarrow \frac{\pi}{4} \left(\log\left(1 + \sqrt{2}\right) \right)$

Q) Evaluate $\int_0^2 \int_0^x e^{x+y}\, dy\, dx.$

$\Rightarrow \int_{x=0}^2 \left[\int_0^x e^y\, dy \right] e^x\, dx.$

$\Rightarrow \int_{x=0}^2 [e^y]_0^x\, e^x\, dx$

$= \int_0^2 (e^x - 1) e^x\, dx = \int_0^2 (e^{2x} - e^x)\, dx$

$1+x^2=a^2$
$\sqrt{1+x^2}=a$

$x^2))\Big|_0^1$

$$= \left[\frac{e^{2x}}{2} - e^x\right]_0^2$$

$$= \frac{e^4}{2} - e^2 - \left[\frac{1}{2} - 1\right]$$

$$= \frac{e^4}{2} - e^2 + \frac{1}{2} = \frac{1}{2}(e^4 - 2e^2 + 1)$$

$$= \frac{1}{2}(e^2 - 1)^2$$

Q) Evaluate $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} \, dx\, dy$.

$$= \int_0^\infty \left[\int_0^\infty e^{-y^2} dy\right] e^{-x^2} dx$$

$$= \int_0^\infty \frac{\sqrt{\pi}}{2} e^{-x^2} dx$$

$$= \frac{\sqrt{\pi}}{2} \int_0^\infty e^{-x^2} dx$$

$$= \frac{\sqrt{\pi}}{2} \cdot \frac{\sqrt{\pi}}{2} = \frac{\pi}{4}$$

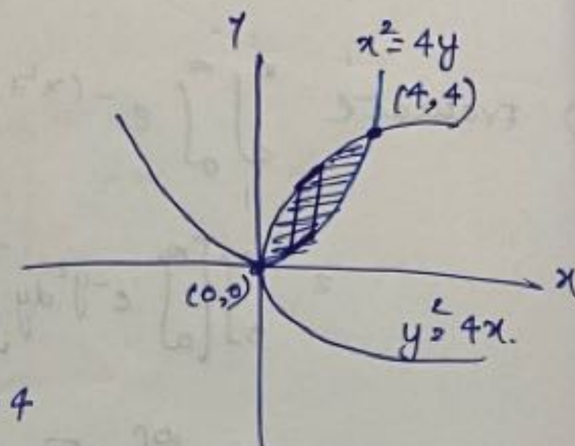
Q) Evaluate $\int_0^4 \int_0^{x^2} e^{y/x} \, dy\, dx$.

" $\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2-y^2} \, dy\, dx$.

Q) Evaluate i) $\iint_R y \, dx\, dy$

ii) $\iint_R y^2 \, dx\, dy$

Where R is the region bounded by the parabola as $y^2 = 4x$ & $x^2 = 4y$



$$\frac{x^4}{4^2} = 4x$$

$$x^4 = 4^3 x \Rightarrow x(x^3 - 4^3) = 0$$

$$\Rightarrow x = 0 \text{ (or) } x = 4$$

when $x = 0 \Rightarrow y = 0$

$$x = 4 \Rightarrow y = \frac{4^2}{4} = 4.$$

$\therefore (0,0), (4,4)$

$\Rightarrow$ Fix $x : 0 \to 4$

$$y : \frac{x^2}{4} \to 2\sqrt{x}$$

$$\Rightarrow \int_0^4 \left[\int_{\frac{x^2}{4}}^{2\sqrt{x}} y \, dy \right] dx$$

$$\Rightarrow \int_0^4 \left[\frac{y^2}{2} \right]_{\frac{x^2}{4}}^{2\sqrt{x}} dx = \int_0^4 \left[\frac{\cancel{4}^{2} x}{\cancel{2}} - \frac{x^4}{32} \right] dx$$

$$\frac{\left(\frac{x^2}{4}\right)^2}{2}$$

the parabola

$$\Rightarrow \int_0^4 \left(2x - \frac{x^4}{32}\right) dx$$

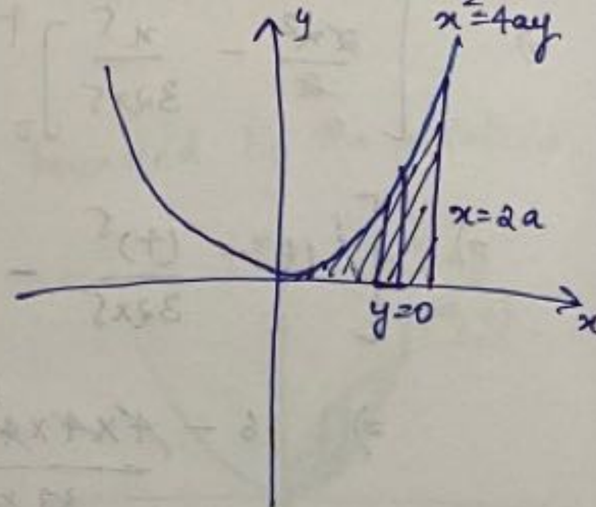
$$\Rightarrow \left[\frac{\cancel{2}x^2}{\cancel{2}} - \frac{x^5}{32\times 5}\right]_0^4$$

$$\Rightarrow \left[16 - \frac{(4)^5}{32\times 5} - 0\right]$$

$$\Rightarrow 16 - \frac{\cancel{4}\times\cancel{4}\times\overset{2}{\cancel{4}}\times 4\times 4}{\underset{\cancel{8}\,\cancel{2}}{\cancel{32}}\times 5}$$

$$\Rightarrow 16 - \frac{32}{5} \Rightarrow \frac{80-32}{5} = \frac{48}{5}$$

Q) Evaluate $\iint xy \, dx\,dy$, where R is the region bounded by x-axis ~~co~~ ordinate $x = 2a$ and the curve $x^2 = 4ay$.



$x = 2a$

$x^2 = 4ay$

$\Rightarrow 4a^2 = 4ay$

$4a(a-y) = 0$

$\Rightarrow y = a$

when $y = a \Rightarrow x = 2a$. On x-axis $y = 0 \Rightarrow x = 0$

$\therefore$ point of intersectⁿ $(0,0)\,(2a,a)$

Fix x, $x : 0 \to 2a$

$y : 0 \to x^2/4a$

$$\iint xy \, dy\,dx = \int_{x=0}^{2a} \left[\int_0^{\frac{x^2}{4a}} y\,dy \right] x\,dx.$$

$$\Rightarrow \int_0^{2a} \left[\frac{y^2}{2} \right]^{x^2/4a} x\,dx$$

$$\Rightarrow \frac{1}{2} \int_0^{2a} \frac{x^4}{16a^2} x\,dx.$$

$$\Rightarrow \frac{1}{32a^2} \int_0^{2a} x^5\,dx$$

$$= \frac{1}{32a^2} \left[\frac{x^6}{6} \right]_0^{2a}$$

$$= \frac{2^6 a^6}{3 \cdot 2\, a^2 \times 6} = \frac{a^4}{3}$$

8) bounded by

when $y = 0 \Rightarrow$

$\Rightarrow x$

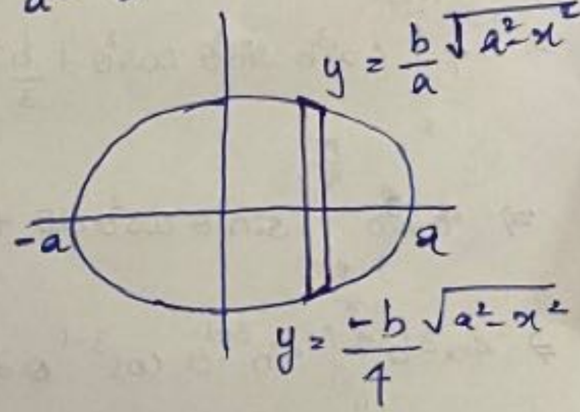
$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$

$y^2 =$

$\Rightarrow \iint (x^2+y^2)$

Pu

8) Evaluate $\iint (x^2+y^2)\,dx\,dy$. over the area bounded by ~~$\frac{x^2}{a^2}$~~ ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$



when $y=0 \Rightarrow \frac{x^2}{a^2}=1$

$\Rightarrow x^2=a^2$

$x=\pm a$, --

$$\frac{y^2}{b^2}=1-\frac{x^2}{a^2}=\frac{a^2-x^2}{a^2}$$

$$y^2=\frac{b^2}{a^2}(a^2-x^2) \Rightarrow y=\pm\frac{b}{a}\sqrt{a^2-x^2}$$

$$\Rightarrow \iint (x^2+y^2)\,dx\,dy = \int_{x=a}^{a}\int_{y=-\frac{b}{a}\sqrt{a^2-x^2}}^{y=\frac{b}{a}\sqrt{a^2-x^2}} (x^2+y^2)\,dx\,dy.$$

$$=2\int_{-a}^{a}\left[\int_{0}^{\frac{b}{a}\sqrt{a^2-x^2}}(x^2+y^2)\,dy\right]dx$$

$$=2\int_{-a}^{a}\left[x^2y+\frac{y^3}{3}\right]_0^{\frac{b}{a}\sqrt{a^2-x^2}}dx.$$

$$=2\int_{-a}^{a}\left[x^2\frac{b}{a}\sqrt{a^2-x^2}+\frac{1}{3}\frac{b^3}{a^3}(a^2-x^2)^{3/2}\right]dx$$

$$=4\int_{0}^{a}\left[x^2\frac{b}{a}(a^2-x^2)^{1/2}+\frac{1}{3}\frac{b^3}{a^3}(a^2-x^2)^{3/2}\right)dx$$

Put $x=a\sin\theta$

$dx=a\cos\theta\,d\theta$

when $x=0 \Rightarrow \theta=0$

$x=a \Rightarrow \theta=\pi/2$

$$\Rightarrow 4\int_0^{\frac{\pi}{2}}\left[a^2\frac{b}{a}\sin^2\theta(a\cos\theta)+\frac{1}{3}\frac{b^3}{a^3}a^3\cos^3\theta\right]a\cos\theta$$

$$\Rightarrow 4\int_0^{\pi/2}\left(a^3b\sin^2\theta\cos^2\theta+\frac{b^3}{3}a\cos^4\theta\right]d\theta.$$

$$\Rightarrow 4a^3b\int_0^{\frac{\pi}{2}}\sin^2\theta\cos^2\theta\,d\theta+\frac{4ab^3}{3}\int_0^{\frac{\pi}{2}}\cos^4\theta\,d\theta.$$

$$\Rightarrow 4a^3b\int_0^{\frac{\pi}{2}}\sin^{3-1}\theta\cos^{3-1}\theta\,d\theta+\frac{4ab^3}{3}\int_0^{\frac{\pi}{2}}\sin^{2(\frac{1}{2})-1}\theta\cos^{5-1}\theta\,d\theta$$

$$\Rightarrow 4a^3b\times\frac{1}{2}\beta(2,2)+\frac{4ab^3}{3}\beta\left(\frac{1}{2},5\right)\times\frac{1}{2}$$

$$\Rightarrow 4a^3b\times\frac{1}{2}\times\frac{\Gamma 2\,\Gamma 2}{\Gamma 4}+\frac{4ab^3}{3}\times\frac{1}{2}\times\frac{\Gamma\frac{1}{2}\,\Gamma 5}{\Gamma\frac{11}{2}}.$$

$$\Rightarrow 4a^3b\times\frac{1}{2}\times\frac{1!\,1!}{3!}+\frac{4ab^3}{3}\times\frac{1}{2}\times\frac{\sqrt{\pi}\;4!}{\frac{9}{2}\cdot\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}$$

$$\Rightarrow \cancel{4}a^3b\times\frac{1}{\cancel{2}}\times\frac{1}{3\times\cancel{2}}+\frac{4ab^3}{3}\times\frac{1}{\cancel{2}}\times\frac{4\times\cancel{3}\times\cancel{2}\times 2\times 2\times 2\times 2\times 2}{9\times 7\times 5\times\cancel{3}}$$

$$\Rightarrow \frac{a^3b}{3}+\frac{4ab^3}{3}\times\frac{4\times 8\times 4}{315}$$

$$\begin{array}{r}\overset{3}{45}\\ \times 7\\ \hline 315\end{array}$$

Q) Evaluate $\iint_R (x^2+y^2)\,dx\,dy$ where R is the region in the +ve quadrant $x+y\le 1$.

Q) Evaluate $\iint y\,dx\,dy$, when R is the domain bounded by y-axis, the curve $y=x^2$ and the line $x+y=2$.

Double Integrals in polar-coordinates:

1) Evaluate $\int_0^{\pi}\int_0^{a\sin\theta} r\,dr\,d\theta$

$$\Rightarrow \int_0^{\pi}\left[\int_{r=0}^{a\sin\theta} r\,dr\right]d\theta = \int_0^{\pi}\left[\frac{r^2}{2}\right]_0^{a\sin\theta} d\theta$$

$$\Rightarrow \frac{1}{2}\int_0^{\pi} a^2\sin^2\theta\,d\theta$$

$$\Rightarrow \frac{a^2}{2}\int_0^{\pi}\sin^2\theta\,d\theta = \frac{a^2}{2}\int_0^{\pi}\frac{1-\cos 2\theta}{2}d\theta$$

$$\Rightarrow \frac{a^2}{4}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi}$$

$$\Rightarrow \frac{a^2}{4}\left[\pi - 0\right] \Rightarrow \frac{\pi a^2}{4}$$

2) $\int_0^{\infty}\int_0^{\frac{\pi}{2}} e^{-r^2} r\,d\theta\,dr$

$$\Rightarrow \int_0^{\infty}\left[\int_0^{\frac{\pi}{2}} d\theta\right]e^{-r^2} r\,dr.$$

$$\Rightarrow \int_0^{\infty}[\theta]_0^{\pi/2} \Rightarrow \frac{\pi}{2}\int_0^{\infty} e^{-r^2} r\,dr \Rightarrow \frac{\pi}{2}\int_0^{\infty} e^{-t}\frac{dt}{2}$$

$$\Rightarrow \frac{\pi}{4}\int_0^{\infty} e^{-t}\,dt = \frac{\pi}{4}\left[e^{-\infty} - e^{-0}\right]$$

$$, \frac{\pi}{4}$$

Q) Evaluate $\int_0^{\frac{\pi}{4}}\int_0^{a\sin\theta} \frac{1}{\sqrt{a^2-r^2}}\, r\,dr\,d\theta$

$$\Rightarrow \int_0^{\frac{\pi}{4}} \left[\int_0^{a\sin\theta} \frac{r}{\sqrt{a^2-r^2}}\,dr\right] d\theta$$

$$\Rightarrow \int_0^{\pi/4} \left[\int_{t=a^2}^{a^2\cos^2\theta} \frac{1}{\sqrt{t}}\left(-\frac{dt}{2}\right)\right] d\theta$$

Put $a^2-r^2=t$

$-2r\,dr = dt$

$r\,dr = \frac{dt}{-2}$

$r=0 \Rightarrow t=a^2$

$r=a\sin\theta$

$\Rightarrow t=a^2-a^2\sin^2\theta$

$=a^2\cos^2\theta$

$$\Rightarrow \int_0^{\pi/4} \left[2\sqrt{t}\left(-\frac{dt}{2}\right)\right] d\theta$$

$$\Rightarrow -\int_0^{\pi/4} \left[\sqrt{a^2\cos^2\theta} - \sqrt{a^2}\right] d\theta$$

$$\Rightarrow -\int_0^{\pi/4} (a\cos\theta - a)\,d\theta$$

$$\Rightarrow -a\int_0^{\pi/4} (\cos\theta - 1)\,d\theta$$

$$\Rightarrow -a\left[\sin\theta - \theta\right]_0^{\pi/4} = -a\left[\sin\frac{\pi}{4} - \frac{\pi}{4}\right]$$

$$= -a\left[\frac{1}{\sqrt{2}} - \frac{\pi}{4}\right]$$

$$= a\left[\frac{\pi}{4} - \frac{1}{\sqrt{2}}\right]$$

Q) Evaluate

$r = a(1-c$

$\Rightarrow \int_0^{\pi}\left[\int_0^{a(1-c}\right.$

$= \int_0^{\pi}\left[\right.$

$= \frac{1}{2}$

$\Rightarrow \frac{a^2}{2}$

$\Rightarrow \frac{a}{2}$

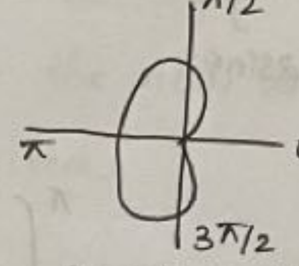
$\Rightarrow 4$

Q) Evaluat

b/w the

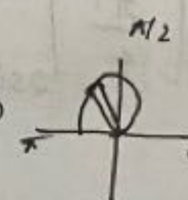
$\oint$

Q) Evaluate $\iint r\sin\theta \, dr\, d\theta$ over the cardiod $r = a(1-\cos\theta)$ above the initial line.

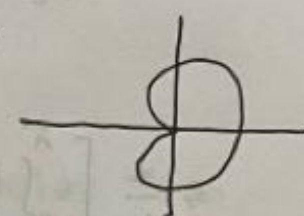


$r = a(1-\cos\theta)$

$$\Rightarrow \int_0^{\pi} \left[\int_0^{a(1-\cos\theta)} r \, dr \right] \sin\theta \, d\theta$$



$$= \int_0^{\pi} \left[\frac{r^2}{2} \right]_0^{a(1-\cos\theta)} \sin\theta \, d\theta \Rightarrow$$


$r = a(1+\cos\theta)$

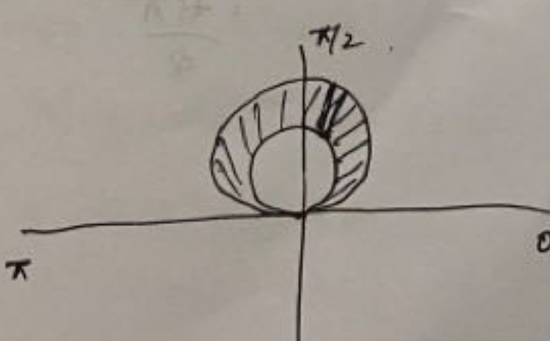
$$= \frac{1}{2} \int_0^{\pi} a^2 (1-\cos\theta)^2 \sin\theta \, d\theta.$$

$$\Rightarrow \frac{a^2}{2} \int_0^{\pi} (1-\cos\theta)^2 \sin\theta \, d\theta \quad \Rightarrow 1-\cos\theta = t$$

$$\Rightarrow \frac{a^2}{2} \int_0^{2} t^2 dt \Rightarrow \frac{a^2}{2} \left[\frac{t^3}{3} \right]_0^2 \Rightarrow \cancel{\frac{a^2}{2} \left[\frac{1}{\sqrt{2}} - \frac{\pi}{4} \right]}$$

$$\Rightarrow \frac{4a^2}{3}. \qquad \Rightarrow \cancel{\frac{a^2}{2} \times \frac{8}{3} \Rightarrow \frac{4a^2}{3}} \qquad \cancel{= a\left[\frac{\pi}{4} - \frac{1}{\sqrt{2}}\right]}$$

Q) Evaluate $\iint r^3 \, dr\, d\theta$ over the area included b/w the circles $r = 2\sin\theta$ and $r = 4\sin\theta$



$$\int_0^{\pi}\int_{2\sin\theta}^{4\sin\theta} r^3\,dr\,d\theta = \int_0^{\pi}\left[\int_{2\sin\theta}^{4\sin\theta} r^3\,dr\right]d\theta$$

$$= \int_0^{\pi}\left[\frac{r^4}{4}\right]_{2\sin\theta}^{4\sin\theta} d\theta$$

$$\Rightarrow \frac{1}{4}\left[\int_0^{\pi}(4^4\sin^4\theta - 2^4\sin^4\theta)\,d\theta\right.$$

$$\Rightarrow \frac{1}{4}\int_0^{\pi} 16\sin^4\theta\,(16-1)\,d\theta.$$

$$\Rightarrow \frac{1}{4}\times 16\times 15\int_0^{\pi}\sin^4\theta\,d\theta$$

$$\Rightarrow \frac{1}{4}\times 16\times 15\times 2\int_0^{\pi/2}\sin^4\theta\,d\theta$$

$(\sin\theta)^4$

$(\sin(-\theta))^4$

$\Rightarrow(-\sin\theta)^4$

$=\sin\theta.$

$$\Rightarrow \frac{1}{\cancel{4}_1}\times\cancel{16}^{4}\times 15\times 2\int_0^{\pi/2}\sin^{2(\frac{5}{2})-1}\theta\,\cos^{2(\frac{1}{2})-1}\theta\,d\theta$$

$$\Rightarrow 120\times\frac{\Gamma\frac{5}{2}\,\Gamma\frac{1}{2}}{\Gamma 3} \Rightarrow 120\times\frac{\frac{3}{2}\times\frac{1}{2}\times\Gamma\frac{1}{2}\,\Gamma\frac{1}{2}}{\Gamma 3}$$

$$= \frac{45\pi}{2}.$$

Q Change

→ Let $x = f$

b/w the

u, v of

$\iint_R F(x$

Note: To ch

put $x=$

Q Evaluate

into polar

$\int_0^a\int_0^{\sqrt{a^2-x}}$

Given y

To change

$x = r\cos\theta$

$x^2 = r^2 c$

$\Rightarrow$

Change of Variables in double integrals.

→ Let $x = f(u, v)$ and $y = g(u, v)$ be the relatⁿ b/w the old variables x, y with the new variables u, v of the new coordinate system.

$$\iint_R F(x, y)\,dx\,dy = \iint F(f, g)\,|J|\,du\,dv.$$

Note: To change the cartesian to polar form,

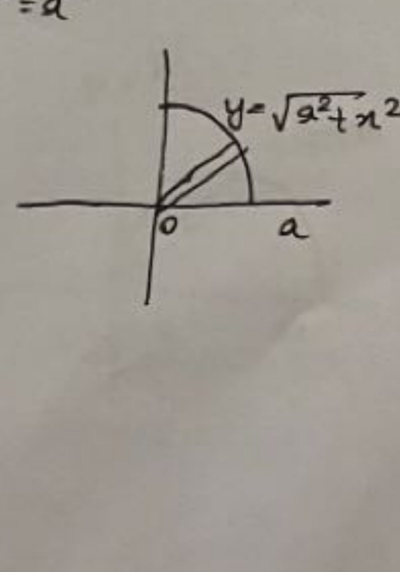
put $x = r\cos\theta$, $y = r\sin\theta$, $dx\,dy = r\,dr\,d\theta$

Evaluate the following integral by transforming into polar coordinates.

$$\int_0^a \int_0^{\sqrt{a^2 - x^2}} y\sqrt{x^2 + y^2}\,dx\,dy$$

Given $y = 0$, $y = \sqrt{a^2 - x^2} \Rightarrow x^2 + y^2 = a^2$

$x = 0$, $x = a$.



To change cartesian to polar put

$x = r\cos\theta \quad y = r\sin\theta$

$x^2 = r^2\cos^2\theta \quad y^2 = r^2\sin^2\theta.$

$\Rightarrow x^2 + y^2 = r^2.$

$\Rightarrow r^2 = a^2 \Rightarrow r = \pm a.$

$$\Rightarrow \int_0^a \int_0^{\sqrt{a^2-x^2}} y\sqrt{x^2+y^2}\,dx\,dy = \int_0^{\frac{\pi}{2}}\int_0^a r\sin\theta\,(r)\,r\,dr\,d\theta$$

$$= \int_0^{\pi/2}\left[\int_0^a r^3\,dr\right]\sin\theta\,d\theta$$

$$= \int_0^{\pi/2}\left[\frac{r^4}{4}\right]_0^a \sin\theta\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{a^4}{4}\sin\theta\,d\theta$$

$$= \frac{a^4}{4}\left[-\cos\theta\right]_0^{\pi/2}$$

$$= -\frac{a^4}{4}\left(\cos\pi/2 - \cos 0\right)$$

$$= \frac{-a^4}{4}(0-1) = \frac{a^4}{4}$$

Q) Tra
and
$\int_0^a$
Q) Eva
Q) Eval
R
0 -
Given
x^2+
θ :
r :

Q) Transform the integral into polar coordinates and hence find the integral.

$$\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{x^2+y^2}\, dx\, dy.$$

Q) Evaluate the double integral ~~θ~~ to

$$\int_0^a \int_0^{\sqrt{a^2-y^2}} (x^2+y^2)\, dx\, dy.$$

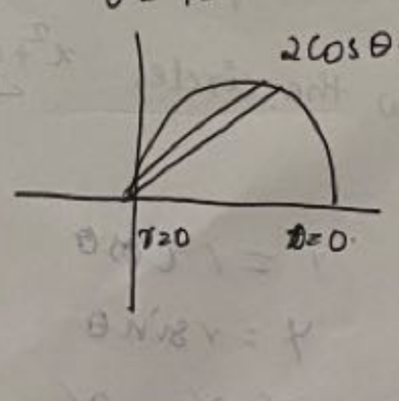
Q) Evaluate D.I ,

$$\int_0^2 \int_0^{\sqrt{2x-x^2}} (x^2+y^2)\, dx\, dy.$$

Given $y=0 \Rightarrow y=\sqrt{2x-x^2}$, $y^2 = 2x - x^2 \Rightarrow x^2+y^2 = 2x$

$x=0 \Rightarrow x=2.$

$x^2+y^2 = r^2(\sin^2\theta + \cos^2\theta) = r^2$



$\theta : 0 \to \frac{\pi}{2}$ $\qquad r^2 = 2r\cos\theta$

$r : 0 \to 2\cos\theta$ $\qquad r = 2\cos\theta$

$$\int_0^2 \int_0^{\sqrt{2x-x^2}} (x^2+y^2)\, dx\, dy = \int_0^{\frac{\pi}{2}} \int_0^{2\cos\theta} r^2\, r\, dr\, d\theta$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \left[\frac{r^4}{4}\right]_0^{2\cos\theta} d\theta.$$

$$\Rightarrow \int_0^{\pi/2} \frac{16\cos^4\theta}{4}\, d\theta$$

$$\Rightarrow 4\int_0^{\frac{\pi}{2}} \cos^4\theta\, d\theta$$

$$\Rightarrow 4\int_0^{\frac{\pi}{2}} \sin^{2\left(\frac{1}{2}\right)-0}\theta \cos^{2\left(\frac{5}{2}\right)-1}\theta\, d\theta$$

$$\Rightarrow 4\,\frac{\Gamma\frac{1}{2}\,\Gamma\frac{5}{2}}{\Gamma 3}$$

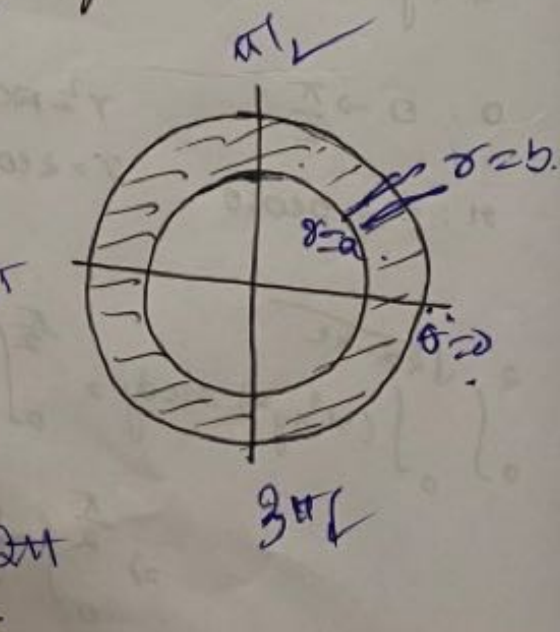
$$\Rightarrow \cancel{4}^{\,2}\times\frac{\Gamma\frac{1}{2}\cdot\frac{3}{\cancel{2}}\cdot\frac{1}{2}\Gamma\frac{1}{2}}{\cancel{2}!}$$

$$\Rightarrow (\sqrt{\pi})^2\times\frac{3}{2} \Rightarrow \frac{3(\sqrt{\pi})^2}{2} \Rightarrow \frac{3\pi}{2}.$$

9) By changing into polar coordinates, evaluate

$$\iint \frac{x^2y^2}{x^2+y^2}\,dx\,dy$$ over the angular region

b/w the circles $x^2+y^2=a^2$, $x^2+y^2=b^2$ $(b>0)$



$x = r\cos\theta$

$y = r\sin\theta$

$x^2+y^2 = r^2$

$r^2 = a^2$

$r = a$

$\theta : 0 \to 2\pi$

$r : a \to b.$

---

i) $\int_0^a \int_0^{\sqrt{a^2-y^2}}$ (

Given.

$x = r\cos\theta$

$r^2\cos^2\theta + r$

$r^2(1) =$

$r =$

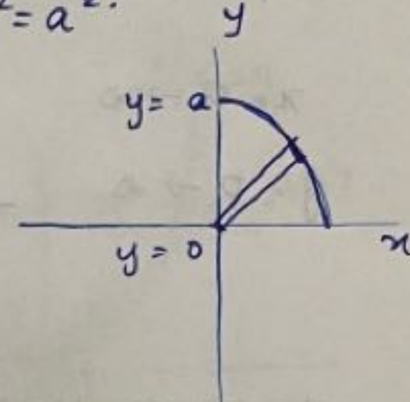
$\Rightarrow \int_0^a \int_0^{\sqrt{a^2-y^2}} (x$

Homework.

2) $\int_0^a \int_0^{\sqrt{a^2-y^2}} (x^2+y^2)\,dx\,dy$.

Given. $x=0 \rightarrow x=\sqrt{a^2-y^2} = x^2+y^2=a^2$.

$y=0 \rightarrow y=a$.



$x = r\cos\theta \quad y = r\sin\theta$.

$r^2\cos^2\theta + r^2\sin^2\theta = a^2$

$r^2(1) = a^2$

$r = \pm a$.

$\Rightarrow \int_0^a \int_0^{\sqrt{a^2-y^2}} (x^2+y^2)\,dx\,dy = \int_0^a \int_0^{\frac{\pi}{2}} r^2\,dr\,d\theta$.

$\Rightarrow \int_0^{\frac{\pi}{2}} \left[\int_0^a r^2\,dr\right] d\theta$

$\Rightarrow \int_0^{\frac{\pi}{2}} \left[\frac{r^3}{3}\right]_0^a d\theta$.

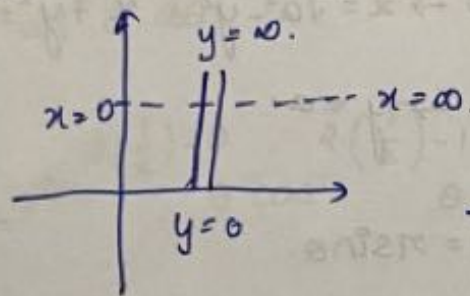
$\Rightarrow \int_0^{\frac{\pi}{2}} \frac{a^3}{3}\,d\theta = \frac{a^3}{3}\left[\theta\right]_0^{\frac{\pi}{2}}$

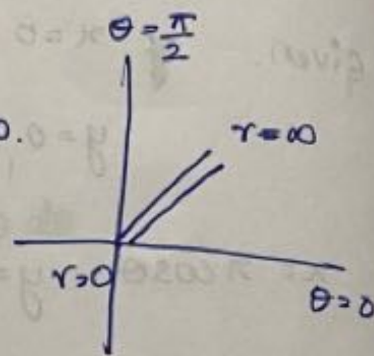
$\Rightarrow \frac{a^3}{3}\left[\frac{\pi}{2}\right] = \frac{a^3\pi}{6}$.

Q) Evaluate $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx\,dy$ by changing to polar coordinates.

$x = 0 \to \infty$

$y = 0 \to \infty$





$x = r\cos\theta \quad y = r\sin\theta.$

$dx\,dy = r\,dr\,d\theta$

$$\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx\,dy = \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} e^{-r^2} r\,dr\,d\theta.$$

$$= \int_{\theta=0}^{\frac{\pi}{2}} \left[\int_{r=0}^{\infty} e^{-r^2} r\,dr \right] d\theta$$

$r^2 = t$

$2r\,dr = dt$

$r\,dr = \frac{dt}{2}$

$r = 0 \Rightarrow t = 0$

$r = \infty \Rightarrow t = \infty$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \left[\frac{1}{2} \int_0^\infty e^{-t} dt \right] d\theta$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \left[\frac{1}{2} \left[-e^{-t} \right]_0^\infty d\theta \right.$$

$$\Rightarrow \frac{1}{2} \int_0^{\frac{\pi}{2}} -1\,d\theta.$$

$$\Rightarrow -\frac{1}{2} [\theta]_0^{\frac{\pi}{2}}$$

$$= -\frac{\pi}{4}.$$

Q CHANGE OF

Q) change the

$\int_0^{4a}$

$y = \frac{x^2}{4a} \Rightarrow$

$y = 2\sqrt{ax}$

$x = 0 \to x =$

$y^2 = 4ax$

$x = \frac{y^2}{4a}$

$x = \frac{4a \times 4a}{4a}$

$x = 4a$

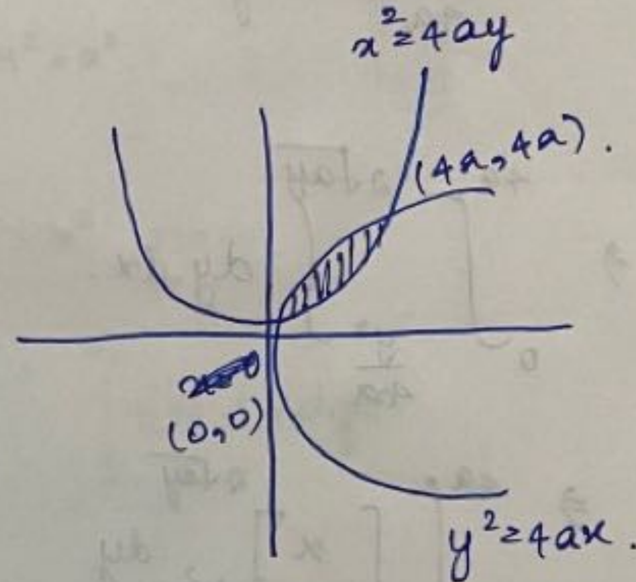
$x = \frac{0}{4a}$

$x = 0.$

CHANGE OF ORDER OF INTEGRATION:

Q) change the order of integration and evaluate

$$\int_0^{4a}\int_{\frac{x^2}{4a}}^{2\sqrt{ax}} dy\,dx$$



$y = \frac{x^2}{4a} \Rightarrow x^2 = 4ay$

$y = 2\sqrt{ax} \Rightarrow y^2 = 4ax$.

$x=0 \rightarrow x = 4a$.

$y^2=4ax$, $x^2 = 4ay$

$x = \frac{y^2}{4a}$ $\quad \left(\frac{y^2}{4a}\right)^2 = 4ay \Rightarrow y^2\left(\frac{y^3}{4a} - 4a\right) = 0.$

$x = \frac{4a\times\cancel{4a}}{\cancel{4a}}$ $\quad \frac{y^{\cancel{4}3}}{16a^2} = 4a\cancel{y}$ $\quad y=0$ or

$\Rightarrow y^3 = 4^3 \times a^3$ $\quad \left(\frac{y^3}{4a} = 4a\right)$

$x = 4a$ $\quad \Rightarrow y = 4a$. $\quad y = 4a$.

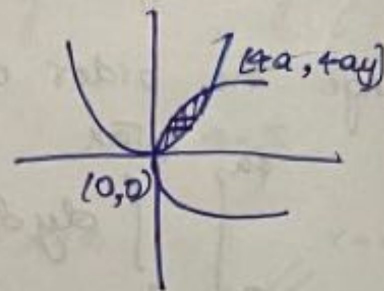
$x = \frac{0}{4a}$

$x = 0$.

To change the order of integrat^n (fix y)

$y : 0 \to 4a$

$x : \frac{y^2}{4a} \to 2\sqrt{ay}$



$$\Rightarrow \int_0^{4a} \int_{\frac{y^2}{4a}}^{2\sqrt{ay}} dy\, dx.$$

$$\Rightarrow \int_0^{4a} \left[x \right]_{\frac{y^2}{4a}}^{2\sqrt{ay}} dy$$

$$\Rightarrow \int_0^{4a} \left(2\sqrt{ay} - \frac{y^2}{4a} \right) dy$$

$$\Rightarrow \left[2\sqrt{a} \frac{y^{3/2}}{3/2} - \frac{y^3}{3 \times 4a} \right]_0^{4a}$$

$$\Rightarrow \left[2\sqrt{a} \times \frac{2}{3} \times (4a)^{3/2} - \frac{(4a)^3}{3 \times 4a} \right]$$

$$\Rightarrow \left[2\sqrt{a} \times \frac{2}{3} \times (2\sqrt{a})^3 - \frac{(4a)^3}{3 \times 4a} \right]$$

$$\Rightarrow \left[(4a)^2 \left(\frac{2}{3} - \frac{4a}{3 \times 4a} \right) \right] = 4(a)^2 \left(\frac{2}{3} - \frac{1}{3} \right) = \frac{4a^2}{3}$$

$$\Rightarrow \cancel{(4a)^2 \left(\frac{8a - 1}{3 \times 4a} \right) \Rightarrow \frac{32a^2 - 4a}{3}}$$

Home Work:

1) $\int_0^a \int_0^{\sqrt{a^2 - x^2}} \sqrt{x^2 + y}$

$\Rightarrow y = 0 \to y = \sqrt{}$

$x = 0 \to x = a$

$x = r\cos\theta,\ y = r$

$\Rightarrow \int_0^a \int_0^{\sqrt{a^2 - x^2}} \sqrt{x^2 + y}$

$\Rightarrow \int_0^{\frac{\pi}{2}} [$

$\Rightarrow \int_0^{\frac{\pi}{2}} [\frac{r}{3}$

2) $\iint \frac{x^2 y^2}{x^2 + y^2}$

$x^2 + y^2 = b^2$

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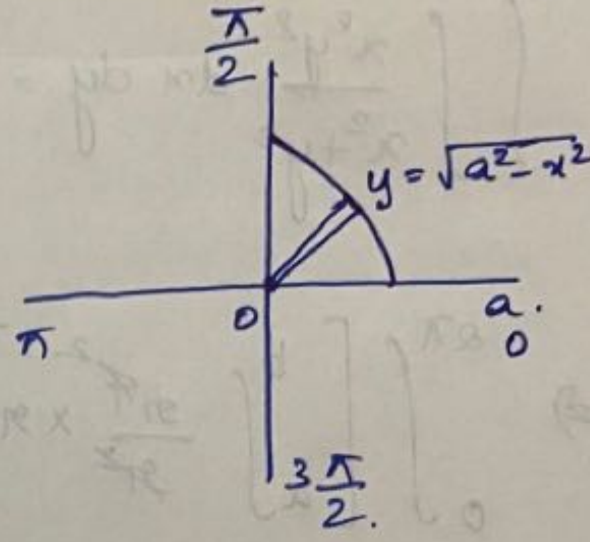
Home Work:

1) $\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{x^2+y^2}\, dx\, dy.$



$\Rightarrow y=0 \rightarrow y=\sqrt{a^2-x^2} \Rightarrow y^2+x^2=a^2.$

$x=0 \rightarrow x=a.$

$x=r\cos\theta,\ y=r\sin\theta,\ x^2+y^2=r^2,\ r=a.$

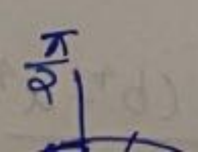
$\Rightarrow \int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{x^2+y^2}\, dx\, dy = \int_0^{\frac{\pi}{2}} \int_0^a r\, r\, dr\, d\theta$

$\Rightarrow \int_0^{\frac{\pi}{2}} \left[\int_0^a [r^2]\, dr \right] d\theta$

$\Rightarrow \int_0^{\frac{\pi}{2}} \left[\frac{r^3}{3}\right]_0^a d\theta \Rightarrow \int_0^{\frac{\pi}{2}} \frac{a^3}{3}\, d\theta$

$\Rightarrow \frac{a^3}{3} \left[\theta\right]_0^{\frac{\pi}{2}} \Rightarrow \frac{\pi a^3}{6}.$

2) $\iint \frac{x^2y^2}{x^2+y^2}\, dx\, dy.$ over the region $x^2+y^2=a^2,$ $x^2+y^2=b^2\ (b>0).$



UNIT-V

CALCULUS OF SEVERAL VARIABLES

Limit of funcⁿ of 2 variables:

A funcⁿ $f(x, y)$ is ~~send~~ & said to tend to the limit l, as (x, y) tends to (a, b), i.e., $x \to a$, $y \to b$, corresponding to any +ve number ϵ there exists a +ve number δ such that,

$$|f(x,y) - l| < \epsilon \text{ whenever } |x-a| \leq \delta, |y-b| \leq \delta$$

It can be written as,

$$\lim_{\substack{x \to a \\ y \to b}} f(x,y) = l$$

(or)

$$\lim_{\substack{y \to b \\ x \to a}} f(x,y) = l.$$

Note: Here, if we want to find out the limit of the funcⁿ we have to check along the path,

$x \to 0$

$y \to 0$

$y \to mx$

$y \to mx^2$

VARIABLES

end & said to tend
to (a, b), i.e.,
any +ve number
ch that,
$-a| \le \delta, |y-b| \le \delta$

out the limit of
ng the path,

Continuity of func^n of 2 variables at a point:

We say that the func^n $f(x, y)$ is continuous at a point then,

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$

Q) Evaluate $\displaystyle \lim_{\substack{x\to 1\\ y\to 2}} \frac{2x^2y}{x^2+y^2+1}$

$$\lim_{x\to 1}\left[\lim_{y\to 2} \frac{2x^2y}{x^2+y^2+1}\right]$$

$$\Rightarrow \lim_{x\to 1}\left[\frac{2x^2(2)}{x^2+(2)^2+1}\right]$$

$$\Rightarrow \lim_{x\to 1} \frac{4x^2}{x^2+5}$$

$$\Rightarrow \frac{4(1)^2}{(1)^2+5} = \frac{4}{6} = \frac{2}{3}$$

$$\lim_{y\to 2}\left[\lim_{x\to 1} \frac{2x^2y}{x^2+y^2+1}\right]$$

$$\Rightarrow \lim_{y\to 2}\left[\frac{2y}{y^2+2}\right]$$

$$\Rightarrow \frac{2(2)}{(2)^2+2} = \frac{4}{6} = \frac{2}{3}.$$

$$\therefore \lim_{\substack{x\to 1\\ y\to 2}} \frac{2x^2y}{x^2+y^2+1} = \frac{2}{3}$$

Q) If $f(x,y) = \dfrac{x-y}{2x+y}$, show that $\lim\limits_{x\to 0}\left\{\lim\limits_{y\to 0} f(x,y)\right\}$

$\neq \lim\limits_{y\to 0}\left\{\lim\limits_{x\to 0} f(x,y)\right\}$.

$$\Rightarrow \lim_{x\to 0}\left\{\lim_{y\to 0}\frac{x-y}{2x+y}\right\}$$

$$\Rightarrow \lim_{x\to 0}\left\{\frac{x-0}{2x+0}\right\} = \lim_{x\to 0}\frac{x}{2x} = \frac{1}{2}.$$

$$\Rightarrow \cancel{\frac{0}{2(0)+0} = 0}.$$

$$\lim_{y\to 0}\left\{\lim_{x\to 0}\frac{x-y}{2x+y}\right\}$$

$$\lim_{y\to 0}\left\{\frac{0-y}{0+y}\right\} = \lim_{y\to 0}\frac{-y}{y} = -1$$

$$\therefore \lim_{x\to 0}\left\{\lim_{y\to 0} f(x,y)\right\} \neq \lim_{y\to 0}\left\{\lim_{x\to 0} f(x,y)\right\}$$

$\therefore$ limit of the funcn doesn't exist at $(0,0)$.

Q) Discuss

$= \begin{cases} \dfrac{2xy}{x^2+y} \\ 0 \end{cases}$

$\Rightarrow \lim\limits_{\substack{x\to 0\\ y\to 0}}$

$\Rightarrow \lim\limits_{y\to 0}\{$

$\Rightarrow$

$\Rightarrow$ Along

$\lim\limits_{\substack{x\to 0\\ y\to}}$

Q) Discuss the continuity of the func^n $f(x,y)$

$$= \begin{cases} \dfrac{2xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases}$$

$$\Rightarrow \lim_{\substack{x\to 0 \\ y\to 0}} f(x,y) = \lim_{\substack{x\to 0 \\ y\to 0}} \frac{2xy}{x^2+y^2}.$$

$$= \lim_{x\to 0} \left\{ \lim_{y\to 0} \frac{2xy}{x^2+y^2} \right\}$$

$$= \lim_{x\to 0} \frac{0}{x^2} = 0. = f(0,0).$$

$$\Rightarrow \lim_{y\to 0} \left\{ \lim_{x\to 0} \frac{2xy}{x^2+y^2} \right\}$$

$$\Rightarrow \lim_{y\to 0} \frac{0}{y^2} = 0 = f(0,0).$$

$\Rightarrow$ Along the line $y = mx$.

$$\lim_{\substack{x\to 0 \\ y\to mx}} \frac{2xy}{x^2+y^2} = \lim_{x\to 0} \left\{ \lim_{y\to mx} \frac{2xy}{x^2+y^2} \right\}$$

$$= \lim_{x\to 0} \left\{ \frac{2m\cancel{x^2}}{\cancel{x^2}(1+m^2)} \right\}$$

$$= \cancel{\lim_{x\to 0}} \Rightarrow \frac{2m}{1+m^2}$$

$\therefore$ ~~Lim~~ $\therefore$ Limit not continuous at $(0,0)$.

For different values of m we are getting different limits.

$\therefore$ $\lim\limits_{\substack{x\to0\\y\to0}} f(x,y)$ is not exist

Q) Examine the continuity at the origin of the funcn defined by

$$f(x,y)=\begin{cases}\dfrac{x^2}{\sqrt{x^2+y^2}}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$$

$$\lim_{\substack{x\to0\\y\to0}} f(x,y) = \lim_{x\to0}\left\{\lim_{y\to0}\frac{x^2}{\sqrt{x^2+y^2}}\right\}$$

$$=\lim_{x\to0}\left\{\frac{x^2}{\sqrt{x^2}}\right\}$$

$$=\lim_{x\to0}\left\{\frac{x^{\cancel{2}}}{\cancel{x}}\right\}$$

$$=\lim_{x\to0} x \Rightarrow 0$$

$$\lim_{y\to0}\left\{\lim_{x\to0}\frac{x^2}{\sqrt{x^2+y^2}}\right\}$$

$$\Rightarrow \lim_{y\to0}\left\{\frac{0}{y}\right\}$$

$$\Rightarrow 0$$

we are getting

exist.

he origin of the

$\frac{x^2}{\ldots + y^2}$, $(x, y) \neq (0,0)$

? , $(x, y) = (0,0)$

iii) Along the $y = mx$.

$$\lim_{x \to 0} \left\{ \lim_{y \to mx} \frac{x^2}{\sqrt{x^2 + m^2 x^2}} \right\}$$

$$\Rightarrow \lim_{x \to 0} \frac{x}{\sqrt{1 + m^2}}$$

$$\Rightarrow 0.$$

$\therefore$ Func^n is continuous.

<u>Partial differentiation:</u>

Let $z = f(x, y)$ be a func^n of two variables x and y then limit $\lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$,

if it exists it is said to be partial derivative of $f(x, y)$ with respect to x. It is denoted by

$\frac{\partial z}{\partial x}$ (or) $\frac{\partial f}{\partial x}$ (or) f_x.

Similarly the partial derivatives of $f(x, y)$ w.r.t to y is $\lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$. It is denoted by $\frac{\partial z}{\partial y}$ or $\frac{\partial f}{\partial y}$ or f_y.

Higher order partial derivatives:

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = f_{xx}.$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = f_{yy}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = f_{xy}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = f_{yx}.$$

Note:

$\Rightarrow f_{xy} = f_{yx}.$

Q) Find first & second order partial derivatives of $ax^2 + 2hxy + by^2$ and verify $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$.

Given, $f(x,y) = ax^2 + 2hxy + by^2$

$$\frac{\partial f}{\partial x} = 2ax + 2hy$$

$$\frac{\partial f}{\partial y} = 2hx + 2by.$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = 2a$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = 2b$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(2hx + 2by) = 2h$$

$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial\right.$

$\therefore \frac{\partial^2 f}{\partial y \partial x}$

Q) If $U = \frac{}{\sqrt{x^2}}$

that $\frac{\partial^2 U}{\partial x^2}$

$\frac{\partial U}{\partial x} = \frac{\partial}{\partial x}(x^2 +$

Similarly,

$\Rightarrow \frac{\partial^2 U}{\partial y^2} = 3y^2($

$\Rightarrow \frac{\partial^2 U}{\partial z^2} = 3z^2($

$= (x^2 +$

tial derivatives of

$$= \frac{\partial^2 f}{\partial y \partial x}.$$

$2h$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(2ax + 2hy) = 2h$$

$$\therefore \quad \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}.$$

8) If $U = \dfrac{1}{\sqrt{x^2+y^2+z^2}}$, $x^2+y^2+z^2 \neq 0$ then prove

that $\dfrac{\partial^2 U}{\partial x^2} + \dfrac{\partial^2 U}{\partial y^2} + \dfrac{\partial^2 U}{\partial z^2} = 0$

$-\frac{5}{2} \times x = -\frac{3}{2}$

$-\frac{5}{2} + 1$

$$\frac{\partial U}{\partial x} = \frac{\partial}{\partial x}(x^2+y^2+z^2)^{-1/2} = -\frac{1}{\cancel{2}}(x^2+y^2+z^2)^{-1/2-1}(\cancel{2}x)$$

$$= -x(x^2+y^2+z^2)^{-3/2}$$

$$\frac{\partial^2 U}{\partial x^2} = -x\frac{\partial}{\partial x}(x^2+y^2+z^2)^{-3/2} + (x^2+y^2+z^2)^{-3/2}(-1)$$

$$= -x\left(-\frac{3}{\cancel{2}}(x^2+y^2+z^2)^{-5/2}(\cancel{2}x)\right) - (x^2+y^2+z^2)^{-3/2}$$

$$= 3x^2(x^2+y^2+z^2)^{-5/2} - (x^2+y^2+z^2)^{-3/2}$$

Similarly,

$$\frac{\partial^2 U}{\partial y^2} = 3y^2(x^2+y^2+z^2)^{-5/2} - (x^2+y^2+z^2)^{-3/2}$$

$$\frac{\partial^2 U}{\partial z^2} = 3z^2(x^2+y^2+z^2)^{-5/2} - (x^2+y^2+z^2)^{-3/2}$$

$$= (x^2+y^2+z^2)^{-5/2}(3z^2 - x^2 - y^2 - z^2)$$

$$= (x^2+y^2+z^2)^{-5/2}(-x^2 - y^2 + 2z^2)$$

$$\frac{\partial^2 U}{\partial x^2} = (x^2+y^2+z^2)^{-5/2}(2x^2-y^2-z^2)$$

$$\frac{\partial^2 U}{\partial y^2} = (x^2+y^2+z^2)^{-5/2}(-x^2-2y^2-z^2)$$

$$\frac{\partial^2 U}{\partial z^2} = (x^2+y^2+z^2)^{-5/2}(-x^2-y^2+2z^2)$$

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} =$$

$$\Rightarrow (x^2+y^2+z^2)^{-5/2}(\cancel{2x^2}-\cancel{y^2}-\cancel{z^2}-\cancel{x^2}-\cancel{2y^2}-\cancel{z^2}-\cancel{x^2}-\cancel{y^2}+\cancel{2z^2})$$

$$\Rightarrow (x^2+y^2+z^2)^{-5/2}(0) = 0$$

$$\therefore \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = 0$$

Q) If $U = \log(x^3+y^3+z^3-3xyz)$ prove that

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 U = \frac{-9}{(x+y+z)^2}$$

$$\frac{\partial}{\partial x} = \frac{1}{x^3+y^3+z^3-3xyz}(3x^2-3yz)$$

$$\frac{\partial}{\partial y} = \frac{1}{x^3+y^3+z^3-3xyz}(3y^2-3xz)$$

$$\frac{\partial}{\partial z} = \frac{1}{x^3+y^3+z^3-3xyz}(3z^2-3xy)$$

$z^2)$

$z^2)$

$z^2)$

$-z^2 - x^2 - y^2 + 2z^2)$

ve that

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right) \Rightarrow$$

$$\Rightarrow \cancel{\left\{\frac{3x^2-3yz+3y^2-3zx+3z^2-3xy}{x^3+y^3+z^3-3xyz}\right\}^x}$$

$$\Rightarrow \left\{\frac{3\,\cancel{(x^2+y^2+z^2-xy-yz-xz)}}{(x+y+z)\cancel{(x^2+y^2+z^2-xy-yz-xz)}}\right\}^x$$

$$\Rightarrow \cancel{\frac{9}{(x+y+z)^2}} \qquad \Rightarrow \frac{3}{(x+y+z)}$$

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u =$$

$$\Rightarrow \frac{\partial}{\partial x}\left(\frac{3}{(x+y+z)}\right)+\frac{\partial}{\partial y}\left(\frac{3}{(x+y+z)}\right)+\frac{\partial}{\partial z}\left(\frac{3}{(x+y+z)}\right)$$

$$\Rightarrow \frac{-3}{(x+y+z)^2}-\frac{3}{(x+y+z)^2}-\frac{3}{(x+y+z)^2}$$

$$\Rightarrow \frac{-9}{(x+y+z)^2}.$$

9) If $x^x y^y z^z = e$, show that at $x=y=z$

$$\frac{\partial^2 z}{\partial x \partial y} = -(x \log ex)^{-1}$$

$$x^x y^y z^z = e$$

$$\log e = \log(x^x y^y z^z).$$

$$x \log x + y \log y + z \log z = 1.$$

$$z \log z = 1 - x\log x - y \log y \rightarrow (1)$$

Differentiate $eq^n (1)$ partially w.r.t to x.

$$\frac{\partial z}{\partial x} \log z + z \times \frac{1}{z} \frac{\partial z}{\partial x} = -\left(1 \log x + \cancel{x} \times \frac{1}{\cancel{x}}\right)$$

$$(1+\log z)\frac{\partial z}{\partial x} = -(1+\log x)$$

$$\frac{\partial z}{\partial x} = \frac{-(1+\log x)}{1+\log z}$$

Similarly

$$\frac{\partial z}{\partial y} = \frac{-(1+\log y)}{1+\log z.}$$

$$\Rightarrow \therefore \frac{\partial}{\partial x}\left(\frac{-(1+\log y)}{1+\log z}\right)$$

$$\Rightarrow -(1+\log y)\left(-\frac{1}{(1+\log z)^2}\left(\frac{1}{z}\right)\frac{dz}{dx}\right)$$

$$= -(1+\log y)\left(-\frac{1}{(1+\log z)^2}\left(\frac{1}{z}\right)\left(\frac{-(1+\log y)}{1+\log z}\right)\right)$$

$$= -(1+\log x)\left(-\frac{1}{(1+\log x)^2}\left(\frac{1}{x}\right)(-1)\right)$$

$$\Rightarrow -(1+\log x)\left(+\frac{1}{x(1+\log x)^2}\right)$$

$$= \frac{-1}{x(1+\log x)}.$$

$$\Rightarrow \cancel{-\log e(x(1+\log x))^{-1}}$$

$$\Rightarrow \frac{-\log e}{x(1+\log x)} \cancel{= \frac{-1}{x}}$$

$$\Rightarrow \frac{-1}{x(\log e+\log x)} = \frac{-1}{x\log ex}$$

$$= -(x\log ex)^{-1}$$

Q) If $u = e^{xyz}$, show that $\frac{\partial^3 u}{\partial x \partial y \partial z} = (1+3xyz+x^2y^2z^2)e^{xyz}$

$$\frac{\partial u}{\partial x} = e^{xyz}(yz) \qquad \frac{\partial u}{\partial z} = e^{xyz}(xy)$$

$$\therefore \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\left(e^{xyz}(xy)\right)\right)$$

$$\Rightarrow \frac{\partial}{\partial x}\left(e^{xyz}(x) + xy\,e^{xyz}(xz)\right)$$

$$= \frac{\partial}{\partial x}\left(x e^{xyz} + x^2yz\,e^{xyz}\right)$$

$$\Rightarrow x\,e^{xyz}(yz) + e^{xyz} + yz(2x)e^{xyz} + x^2yz\,e^{xyz}(yz)$$

$$\Rightarrow xyz\,e^{xyz} + e^{xyz} + e^{xyz}\,2xyz + e^{xyz}x^2y^2z^2$$

$$\Rightarrow e^{xyz}(1+3xyz+x^2y^2z^2)$$

Q) If

$\frac{\partial z}{\partial x} = f$

$\frac{\partial^2 z}{\partial x^2} = f$

$\frac{\partial z}{\partial y} = f'$

$\frac{\partial^2 z}{\partial y^2} =$

$= a$

$\frac{\partial^2 z}{\partial y^2} =$

Q) If $U = l$

$(x^2+$

Q) If $V =$

Q) Verify

Q) If z

Q) If $z = f(x+ay) + \phi(x-ay)$, P.T $\frac{\partial^2 z}{\partial x^2} = a^2 \frac{\partial^2 z}{\partial y^2}$.

$$\frac{\partial z}{\partial x} = f'(x+ay)(1) + \phi'(x-ay)(1)$$

$$\frac{\partial^2 z}{\partial x^2} = f''(x+ay) + \phi''(x-ay)$$

$$\frac{\partial z}{\partial y} = f'(x+ay)a + \phi'(x-ay)(-a)$$

$$\frac{\partial^2 z}{\partial y^2} = f''(x+ay)a^2 + \phi''(x-ay)(a^2)$$

$$= a^2(f''(x+ay) + \phi''(x-ay))$$

$$\frac{\partial^2 z}{\partial y^2} = a^2 \frac{\partial^2 z}{\partial x^2}$$

Q) If $v = \log(x^2+y^2+z^2)$, prove that

$$(x^2+y^2+z^2)\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) = 2.$$

Q) If $V = \tan^{-1}\left(\frac{2xy}{x^2-y^2}\right)$, prove that $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$

Q) Verify $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$ for the func^n $u = \tan^{-1}(x/y)$

Q) If $z = \log(e^x + e^y)$, s.T $rt - s^2 = 0$

$$r = \frac{\partial^2 z}{\partial x^2},\quad t = \frac{\partial^2 z}{\partial y^2},\quad s = \frac{\partial^2 z}{\partial x \partial y}.$$

The chain rule of a partial differentiation:

i) If $z = f(x,y)$, where $x = \phi(t)$, $y = \psi(t)$

then z is called composite func^n of a variable t.

ii) $z = f(x,y)$ where $x = \phi(u,v)$, $y = \psi(u,v)$

then z is called a composite func^n of two variables u and v.

Note: $z = f(u,v)$ where $u = \phi(x,y)$, $v = \psi(x,y)$

In this case we can calculate,

$\rightarrow$ $$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}$$

$\rightarrow$ $$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}.$$

Total differential coefficient:

$\rightarrow$ $z = f(x,y)$ $\quad x = \phi(t)$, $y = \phi(t)$

* $$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$

8) If $u =$

S.T x

$\frac{\partial u}{\partial x} = \frac{\partial}{\partial}$

$\frac{\partial u}{\partial y} = \frac{\partial}{\partial}$

$\frac{\partial u}{\partial z} = \frac{\partial}{\partial}$

$r = \frac{x}{y}$

$\frac{\partial r}{\partial x} = \frac{1}{y}$

$\frac{\partial r}{\partial y} = -\frac{x}{y}$

$\frac{\partial r}{\partial z} = 0$

$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}$

$\frac{\partial u}{\partial y} = \frac{\partial}{\partial}$

$\frac{\partial u}{\partial z} = \frac{\partial}{\partial}$

rentiation:

, $y = \cancel{\phi}\, \psi(t)$

n of a variable t.

$y = \psi(u, v)$

func^n of two

$y)$, $v = \psi(x, y)$

Q) If $u = f(r, s, t)$, where $r = \frac{x}{y}$, $s = \frac{y}{z}$, $t = \frac{z}{x}$

S.T $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$.

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial x} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial x}.$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial y}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial z} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial z}.$$

$r = \frac{x}{y}$ $\quad s = \frac{y}{z}$ $\quad t = \frac{z}{x}$

$\frac{dr}{\partial x} = \frac{1}{y}$ $\quad \frac{\partial s}{\partial x} = 0$ $\quad \frac{\partial t}{\partial x} = \frac{-z}{x^2}$

$\frac{\partial r}{\partial y} = \frac{-x}{y^2}$ $\quad \frac{\partial s}{\partial y} = \frac{1}{z}$ $\quad \frac{\partial t}{\partial y} = 0$

$\frac{\partial r}{\partial z} = 0$ $\quad \frac{\partial s}{\partial z} = \frac{-y}{z^2}$ $\quad \frac{\partial t}{\partial z} = \frac{1}{x}.$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\left(\frac{1}{y}\right) + \frac{\partial u}{\partial t}\left(-\frac{z}{x^2}\right) + \frac{\partial u}{\partial s}(0)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\left(\frac{-x}{y^2}\right) + \frac{\partial u}{\partial s}\left(\frac{1}{z}\right) + \frac{\partial u}{\partial t}(0)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}(0) + \frac{\partial u}{\partial s}\left(-\frac{y}{z^2}\right) + \frac{\partial u}{\partial t}\left(\frac{1}{x}\right)$$

$$\Rightarrow x\frac{\partial u}{\partial x} = \frac{x}{y}\frac{\partial u}{\partial r} - \frac{z}{x}\frac{\partial u}{\partial t}$$

$$\Rightarrow y\frac{\partial u}{\partial y} = \frac{-x}{y}\frac{\partial u}{\partial r} + \frac{y}{z}\frac{\partial u}{\partial s}.$$

$$\Rightarrow z\frac{\partial u}{\partial z} = \frac{-y}{z}\frac{\partial u}{\partial s} + \frac{z}{x}\frac{\partial u}{\partial t}$$

$$\therefore \frac{x\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = \cancel{\frac{x}{y}\frac{\partial u}{\partial r}} - \cancel{\frac{z}{x}\frac{\partial u}{\partial t}}$$

$$+ \cancel{\frac{y}{z}\frac{\partial u}{\partial s}} - \cancel{\frac{x}{y}\frac{\partial u}{\partial r}} - \cancel{\frac{y}{z}\frac{\partial u}{\partial s}} + \cancel{\frac{z}{x}\frac{\partial u}{\partial t}}$$

$$\therefore \frac{x\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0.$$

Q) If $u = f(y-z, z-x, x-y)$, S.T $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$

let $r = y-z$, $s = z-x$, $t = x-y$.

$\cancel{\frac{\partial r}{\partial x} = 0}$ $\cancel{\frac{\partial r}{\partial}}$

$\frac{\partial r}{\partial x} = 0$ $\quad \frac{\partial s}{\partial x} = -1$ $\quad \frac{\partial t}{\partial x} = 1$

$\frac{\partial r}{\partial y} = 1$ $\quad \frac{\partial s}{\partial y} = 0$ $\quad \frac{\partial t}{\partial y} = -1$

$\frac{\partial r}{\partial z} = -1$ $\quad \frac{\partial s}{\partial z} = 1$ $\quad \frac{\partial t}{\partial z} = 0$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial x}$$

$$= \frac{\partial u}{\partial r}(0) + \frac{\partial u}{\partial s}(-1) + \frac{\partial u}{\partial t}(1)$$

$$\frac{\partial u}{\partial x} = -\frac{\partial u}{\partial s} + \frac{\partial u}{\partial t}$$

$$\frac{\partial u}{\partial y} = \cancel{\frac{\partial u}{\partial y}}\ \cancel{\partial y}\ \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial z}$$

$$= \frac{\partial u}{\partial r}(1) + \frac{\partial u}{\partial s}(0) + \frac{\partial u}{\partial t}(-1)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r} - \frac{\partial u}{\partial t}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial z} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial z}$$

$$= \frac{\partial u}{\partial r}(-1) + \frac{\partial u}{\partial s}(1) + \frac{\partial u}{\partial t}(0)$$

$$\frac{\partial u}{\partial z} = -\frac{\partial u}{\partial r} + \frac{\partial u}{\partial s}$$

$$\therefore \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = -\cancel{\frac{\partial u}{\partial s}} + \cancel{\frac{\partial u}{\partial t}} + \cancel{\frac{\partial u}{\partial r}} - \cancel{\frac{\partial u}{\partial t}}$$
$$- \cancel{\frac{\partial u}{\partial r}} + \cancel{\frac{\partial u}{\partial s}}$$

$$\therefore \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0.$$

Homework:

Q) $U = \log(x^2+y^2+z^2)$, p.T $(x^2+y^2+z^2)\left(\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}+\frac{\partial^2 U}{\partial z^2}\right) = 2$

$$\frac{\partial u}{\partial x} = \frac{1}{x^2+y^2+z^2}(2x) \Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{x^2+y^2+z^2(2) - 2x(2x)}{(x^2+y^2+z^2)^2}$$

$$= \frac{2x^2+2y^2+2z^2-4x^2}{(x^2+y^2+z^2)^2}$$

$$= \frac{-2x^2+2y^2+2z^2}{(x^2+y^2+z^2)^2}$$

∴ Similarly,

$$\frac{\partial^2 u}{\partial y^2} = \frac{2x^2-2y^2+2z^2}{(x^2+y^2+z^2)^2} \qquad \frac{\partial^2 u}{\partial z^2} = \frac{2x^2+2y^2-2z^2}{(x^2+y^2+z^2)^2}$$

$$\therefore \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2} = \frac{-\cancel{2x^2}+\cancel{2y^2}+\cancel{2z^2}+\cancel{2x^2}-\cancel{2y^2}+2z^2+2x^2+2y^2-\cancel{2z^2}}{(x^2+y^2+z^2)^2}$$

$$= \frac{2x^2+2y^2+2z^2}{(x^2+y^2+z^2)^2}$$

$$= \frac{2\cancel{(x^2+y^2+z^2)}}{(x^2+y^2+z^2)^{\cancel{2}}}$$

$$= \frac{2}{x^2+y^2+z^2}$$

$$(x^2+y^2+z^2)\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right) = \cancel{x^2+y^2+z^2} \times \frac{2}{\cancel{x^2+y^2+z^2}} = 2.$$

$y^3)$

$2)$

$\rightarrow t$

$2x^2y)$

$^3 + 12x^2y^3$

$8x^3y^3/t$

d) $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$ for $u = \tan^{-1}(x/y)$

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\left(\tan^{-1}(x/y)\right)\right) = \frac{\partial}{\partial x}\left(\frac{1}{1+\left(\frac{x}{y}\right)^2}\left(-\frac{x}{y^2}\right)\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{-x}{y^2+x^2}\right)$$

$$= \frac{(x^2+y^2)(-1)+x(2x)}{(x^2+y^2)^2}$$

$$= \frac{-x^2-y^2+2x^2}{(x^2+y^2)^2}$$

$$= \frac{x^2-y^2}{(x^2+y^2)^2}.$$

$$\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\left(\tan^{-1}(x/y)\right)\right) = \frac{\partial}{\partial y}\left(\frac{1}{1+\frac{x^2}{y^2}} \times \left(\frac{1}{y}\right)\right)$$

$$= \frac{\partial}{\partial y}\left(\frac{1}{y+\frac{x^2}{y}}\right)$$

$$= \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right)$$

$$= \frac{x^2+y^2(1) - y(2y)}{(x^2+y^2)^2}$$

$$= \frac{x^2+y^2-2y^2}{(x^2+y^2)^2}$$

$$= \frac{x^2-y^2}{(x^2+y^2)^2}$$

$\therefore \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}.$

Q) $z = \log(e^x + e^y)$, S.T $rt - s^2 = 0$.

$$r = \frac{\partial^2 z}{\partial x^2},\quad t = \frac{\partial^2 z}{\partial y^2},\quad s = \frac{\partial^2 z}{\cancel{\partial z^2}\,\partial x \partial y}$$

$$\frac{\partial z}{\partial x} = \frac{1}{e^x + e^y}(e^x) \Rightarrow \frac{\partial^2 z}{\partial x^2} = \frac{(e^x + e^y)(e^x) - e^x(e^x)}{(e^x + e^y)^2}$$

$$= \frac{\cancel{e^{2x}} + e^{x+y} - \cancel{e^{2x}}}{(e^x + e^y)^2}$$

$$= \frac{e^{x+y}}{(e^x + e^y)^2}$$

$$\frac{\partial z}{\partial y} = \frac{1}{e^x + e^y}(e^y) \Rightarrow \frac{\partial^2 z}{\partial y^2} = \frac{(e^x + e^y)(e^y) - e^y(e^y)}{(e^x + e^y)^2}$$

$$= \frac{\cancel{e^{x+y}} + \cancel{e^{2y}} - \cancel{e^{2y}} + e^{x+y}}{(e^x + e^y)^2}$$

$$= \frac{e^{x+y}}{(e^x + e^y)^2}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\left(\log(e^x + e^y)\right)\right) = \frac{\partial}{\partial x}\left(\frac{1}{e^x + e^y}(e^y)\right)$$

$$= \frac{(e^x + e^y)(0) - e^y(e^x)}{(e^x + e^y)^2}$$

$$= \frac{0 - e^{x+y}}{(e^x + e^y)^2}$$

$$= \frac{-e^{x+y}}{(e^x + e^y)^2}$$

$$r = \frac{e^{x+y}}{(e^x+e^y)^2}, \quad t = \frac{e^{x+y}}{(e^x+e^y)^2}, \quad s^2 = \frac{(e^{x+y})^2}{(e^x+e^y)^4}$$

$$rt - s^2 = 0$$

$$\therefore \left(\frac{e^{x+y}}{(e^x+e^y)^2}\right)\left(\frac{e^{x+y}}{(e^x+e^y)^2}\right) - \frac{(e^{x+y})^2}{(e^x+e^y)^4}$$

$$\Rightarrow \frac{(e^{x+y})^2}{(e^x+e^y)^4} - \frac{(e^{x+y})^2}{(e^x+e^y)^4} = 0.$$

$$\therefore rt - s^2 = 0$$

Q) If $u = f(r)$ and $x = r\cos\theta$, $y = r\sin\theta$

P.T $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r} f'(r)$

$u = f(r) \Rightarrow \frac{\partial u}{\partial r} = f'(r)$

$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} = f'(r)\frac{\partial r}{\partial x}$

$$\frac{\partial^2 u}{\partial x^2} = f''(r)\left(\frac{\partial r}{\partial x}\right)^2 + f'(r)\frac{\partial^2 r}{\partial x^2}$$

Similarly, $\frac{\partial^2 u}{\partial y^2} = f''(r)\left(\frac{\partial r}{\partial y}\right)^2 + f'(r)\left(\frac{\partial^2 r}{\partial y^2}\right)$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r)\left[\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2\right] + f'(r)\left[\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2}\right]$$

$x = r\cos\theta \quad y = r\sin\theta$

$x^2 + y^2 = r^2$

$\cancel{2}r\frac{\partial r}{\partial x} = \cancel{2}x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r} \Rightarrow \cancel{\frac{\partial^2 r}{\partial x^2} = \frac{1}{r}}$

Similarly, $\frac{\partial r}{\partial y} = \frac{y}{r} \Rightarrow \cancel{\frac{x}{r} = \frac{\partial^2 r}{\partial y}}$

$\Rightarrow f''(r)\left[\frac{x^2}{r^2} + \frac{y^2}{r^2}\right] + f'(r)\left[\frac{1}{r} + \frac{1}{r}\right]$

$\Rightarrow f''(r)\left[\frac{r^2}{r^2}\right] + f'(r)\left[\frac{2}{r}\right]$

$\Rightarrow f''(r) + \frac{1}{r}f'(r)$

$$\frac{\partial^2 r}{\partial x^2} = \frac{d}{\partial x}\left(\frac{x}{r}\right) = \frac{r(1) - x\left(\frac{\partial r}{\partial x}\right)}{r^2} = \frac{r - x\frac{\partial r}{\partial x}}{r^2}$$

$$\frac{\partial^2 r}{\partial y^2} = \frac{r - y\frac{\partial r}{\partial y}}{r^2}.$$

$$\therefore \frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} = \frac{r - x\frac{\partial r}{\partial x} + r - y\frac{\partial r}{\partial y}}{r^2}$$

$$= \frac{r - \frac{x^2}{r} + r - \frac{y^2}{r}}{r^2}$$

$$= \frac{r^2 - x^2 + r^2 - y^2}{r^3}$$

$$= \frac{2r^2 - (x^2 + y^2)}{r^3}$$

$$= \frac{2r^2 - r^2}{r^3} = \frac{\cancel{r^2}}{\cancel{r^3}} = \frac{1}{r}.$$

$$\Rightarrow f''(r)\left[\frac{x^2}{r^2} + \frac{y^2}{r^2}\right] + f'(r)\left[\frac{1}{r}\right]$$

$$\Rightarrow f''(r)\left[\frac{r^2}{r^2}\right] + f'(r)\left[\frac{1}{r}\right]$$

$$\Rightarrow f''(r) + \frac{1}{r}f'(r) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}.$$

Q) If $u = x \log xy$, where $x^2+y^2+3xy = 1$ find $\frac{du}{dx}$.

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx}.$$

$$u = x\log x + x\log y$$

$$\frac{\partial u}{\partial x} = x\frac{1}{x} + 1\log x + \log y$$

$$= 1 + \log(xy)$$

$$\frac{\partial u}{\partial y} = \frac{x}{y}.$$

$$x^2+y^2+3xy = 1.$$

$$2x + 2y\frac{dy}{dx} + \cancel{2y}\,3\left[x\frac{dy}{dx} + y\right] = 0$$

$$\cancel{2y\frac{dy}{dx} = -3y-2x.}$$

$$\cancel{\frac{dy}{\partial x} = \frac{-3y-2x}{2y}.}$$

$$\Rightarrow \quad 2x + 2y\frac{dy}{dx} + 3x\frac{dy}{dx} + 3y = 0$$

$$2x+3y+\frac{dy}{dx}(2y+3x) = 0$$

$$\frac{dy}{dx} = \frac{-2x-3y}{3x+2y}$$

$$\frac{du}{dx} = 1+\log(xy) + \frac{x}{y}\left(\frac{-2x-3y}{3x+2y}\right)$$

$y = 1$ find $\frac{dy}{dx}$

Q) If $u = x\log xy$, where $x^3+y^3+3xy = 1$ find $\frac{du}{dx}$.

<u>Jacobian</u>: Let $u = u(x,y)$, $v = v(x,y)$ then these two simultaneous relatⁿs constitue a transformatⁿ from (x,y) to (u,v).

The determinant $\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$ is called the

Jacobian of (u,v) w.r.t (x,y).

The determinant value is denoted by $\frac{\partial(u,v)}{\partial(x,y)}$ (or)

$J\left(\frac{u,v}{x,y}\right)$.

If $u = u(x,y,z)$, $v = v(x,y,z)$, $w = w(x,y,z)$

then,

$$J\left(\frac{u,v,w}{x,y,z}\right) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}.$$

Properties:

$\rightarrow$ $$\frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(u,v)} = 1$$

$\rightarrow$ Chain rule for Jacobian:

If (u,v) are func^ns of (r,s) and (r,s) are funcns of (x,y), then:

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(r,s)}\,\frac{\partial(r,s)}{\partial(x,y)}$$

$$\Rightarrow J\left(\frac{u,v}{x,y}\right) = \frac{\partial(u,v)}{\partial(r,s)}\,\frac{\partial(r,s)}{\partial(x,y)}$$

Q) If $u = \frac{yz}{x}$, $v = \frac{zx}{y}$, $w = \frac{xy}{z}$, S.T

$$J\left(\frac{u,v,w}{x,y,z}\right) = 4$$

$$J\left(\frac{u,v,w}{x,y,z}\right) = \begin{vmatrix} \frac{-yz}{x^2} & \frac{z}{x} & \frac{y}{x} \\ \frac{z}{y} & \frac{-zx}{y^2} & \frac{x}{y} \\ \frac{y}{z} & \frac{x}{z} & \frac{-xy}{z^2} \end{vmatrix}$$

$$\Rightarrow \frac{-yz}{x^2}\left(\frac{x^2\cancel{yz}}{y^2z^2} - \frac{x^2}{yz}\right) - \frac{z}{x}\left(-\frac{xy\cancel{z}}{yz^{\cancel{2}}} - \frac{x\cancel{y}}{\cancel{y}z}\right)$$

$$+ \frac{y}{x}\left(\frac{x\cancel{z}}{y\cancel{z}} + \frac{xy\cancel{z}}{\cancel{z}y^2}\right)$$

$$= \frac{-yz}{x^2}(0) - \frac{z}{x}\left(-\frac{x}{z} - \frac{x}{z}\right)$$

$$+ \frac{y}{x}\left(\frac{x}{y} + \frac{x}{\cancel{z}y}\right)$$

$$= -\frac{\cancel{z}}{\cancel{x}}\left(-\frac{2\cancel{x}}{\cancel{z}}\right) + \frac{\cancel{y}}{\cancel{x}}\left(\frac{2\cancel{x}}{\cancel{y}}\right)$$

$$= 2 + 2$$

$$= 4$$

$$\Rightarrow J\left(\frac{u, v, w}{x, y, z}\right) = 4.$$

Q) If $x+y+z=u$, $y+z=uv$, $z=uvw$

then evaluate $\frac{\partial(x,y,z)}{\partial(u,v,w)}$, $\frac{\partial(u,v,w)}{\partial(x,y,z)}$

$y+z=uv$

$y+uvw=uv$

$\Rightarrow y=uv-uvw.$

$x+y+z=u$

$x+uv-\cancel{uvw}+\cancel{uvw}=u$

$\Rightarrow x=u-uv$

$$\therefore \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} 1-v & -u & 0 \\ v-vw & u-uw & -uv \\ vw & uw & uv \end{vmatrix}$$

$\Rightarrow 1-v(uv(u-uw)+uv(uw))+u(uv(v-vw)+vw(uv))$

$= 1-v(u^2v-\cancel{u^2vw}+\cancel{u^2vw})+u(uv^2-\cancel{uv^2w}+\cancel{uwv^2})$

$\Rightarrow u^2v-\cancel{u^2v^2}+\cancel{u^2v^2}$

$\Rightarrow u^2v$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = u^2v$$

$\Rightarrow uv = y$

$(x+y+z)$

$\boxed{v = \frac{}{x}}$

$\boxed{u = x+y}$

$\frac{\partial u}{\partial x} = 1$

$\frac{\partial v}{\partial x} = \frac{(x}{}$

$\frac{\partial v}{\partial y} = \frac{(}{}$

$\frac{\partial v}{\partial z} = \frac{(x}{}$

$\frac{\partial w}{\partial x} = \frac{(y-}{}$

$\frac{\partial w}{\partial y} = \frac{(}{}$

$\frac{\partial w}{\partial z} = \frac{(y}{}$

$\Rightarrow uv = y+z$ $\qquad$ $z = uvw$

$(x+y+z)v = y+z$ $\qquad$ $z = \cancel{(x+y+z)}\dfrac{(y+z)}{\cancel{(x+y+z)}}\, w$

$$\boxed{v = \frac{y+z}{x+y+z}} \qquad \boxed{w = \frac{z}{y+z}}$$

$$\boxed{u = x+y+z}$$

$$\frac{\partial u}{\partial x} = 1 \qquad \frac{\partial u}{\partial y} = 1 \qquad \frac{\partial u}{\partial z} = 1.$$

$$\frac{\partial v}{\partial x} = \frac{(x+y+z)(0) - (y+z)(1)}{(x+y+z)^2} = \frac{-(y+z)}{(x+y+z)^2}$$

$$\frac{\partial v}{\partial y} = \frac{(x+y+z)(1) - (y+z)(1)}{(x+y+z)^2} = \frac{x+\cancel{y}+\cancel{z}-\cancel{y}-\cancel{z}}{(x+y+z)^2} = \frac{x}{(x+y+z)^2}$$

$$\frac{\partial v}{\partial z} = \frac{(x+y+z)(1) - (y+z)(1)}{(x+y+z)^2} = \frac{x}{(x+y+z)^2}$$

$$\frac{\partial w}{\partial x} = \frac{(y+z)(0) - z(0)}{(y+z)^2} = 0$$

$$\frac{\partial w}{\partial y} = \frac{(y+z)(0) - z(1)}{(y+z)^2} = \frac{-z}{(y+z)^2}$$

$$\frac{\partial w}{\partial z} = \frac{(y+z)(1) - z(1)}{(y+z)^2} = \frac{y}{(y+z)^2}.$$

$$J\left(\frac{u,v,w}{x,y,z}\right) = \begin{vmatrix} 1 & 1 & 1 \\ \frac{-(y+z)}{(x+y+z)^2} & \frac{x}{(x+y+z)^2} & \frac{x}{(x+y+z)^2} \\ 0 & \frac{-z}{(y+z)^2} & \frac{y}{(y+z)^2} \end{vmatrix}$$

$$1\left(\frac{xy}{(x+y+z)^2(y+z)^2} + \frac{xz}{(y+z)^2(x+y+z)^2}\right) - 1\left(\frac{-y(y+z)}{(x+y+z)^2(y+z)^2}\right) + 1\left(\frac{(y+z)z}{(x+y+z)^2(y+z)^2}\right)$$

$$\Rightarrow \frac{xy+xz}{(x+y+z)^2(y+z)^2} + \frac{y(y+z)}{(x+y+z)^2(y+z)^2} + \frac{z(y+z)}{(x+y+z)^2(y+z)^2}.$$

$$\Rightarrow \frac{xy+xz+y^2+yz+yz+z^2}{(x+y+z)^2(y+z)^2}$$

$$\Rightarrow \frac{x(y+z)+(y+z)^2}{(x+y+z)^2(y+z)^2}$$

$$\Rightarrow \frac{(y+z)(x+y+z)}{(x+y+z)^{\cancel{2}}(y+z)^{\cancel{2}}}$$

$$J\left(\frac{u,v,w}{x,y,z}\right) = \frac{1}{(y+z)(x+y+z)}.$$

$\frac{x}{x+y+z)^2}$

$\frac{y}{y+z)^2}$

$\frac{y+z)}{+z)^2(y+z)^2}$.

8) If $x = r\cos\theta$, $y = r\sin\theta$ find $\frac{\partial(x,y)}{\partial(r,\theta)}$ and $\frac{\partial(r,\theta)}{\partial(x,y)}$ & S.T $\frac{\partial(x,y)}{\partial(r,\theta)} \frac{\partial(r,\theta)}{\partial(x,y)} = 1$.

$$\therefore J\left(\frac{x,y}{r,\theta}\right) = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta = r.$$

$\Rightarrow x^2+y^2 = r^2$

$\frac{y}{x} = \tan\theta \Rightarrow \theta = \tan^{-1}\left(\frac{y}{x}\right)$

$$J\left(\frac{r,\theta}{x,y}\right) = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{vmatrix}$$

$\frac{\partial r}{\partial x} \Rightarrow \cancel{2}r\frac{\partial r}{\partial x} = \cancel{2}x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$.

$\frac{\partial r}{\partial y} = \frac{y}{r}$.

$$\frac{\partial\theta}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{-y}{x^2}\right) = \frac{1}{\frac{y^2+x^2}{x^2}}\left(\frac{-y}{x^2}\right) = \frac{-y}{x^2+y^2} = \frac{-y}{r^2}$$

$$= -\frac{r\sin\theta}{r^2}$$

$$\frac{\partial\theta}{\partial x} = \frac{-\sin\theta}{r}.$$

$$\frac{\partial\theta}{\partial y} = \frac{1}{(1)+\frac{y^2}{x^2}}\left(\frac{1}{x}\right) = \frac{x^2}{x^2+y^2}\left(\frac{1}{x}\right) = \frac{x}{r^2} = \frac{r\cos\theta}{r^2} = \frac{\cos\theta}{r}$$

$$J\left(\frac{r,\theta}{x,y}\right) = \begin{vmatrix} \frac{x}{r} & \frac{y}{r} \\ -\frac{\sin\theta}{r} & \frac{\cos\theta}{r} \end{vmatrix}$$

$$= \frac{x\cos\theta}{r^2} + \frac{y\sin\theta}{r^2}$$

$$= \frac{r\cos^2\theta + r\sin^2\theta}{r^2}$$

$$= \frac{r}{r^2} = \frac{1}{r}$$

$$\therefore J\left(\frac{x,y}{r,\theta}\right) J\left(\frac{r,\theta}{x,y}\right) = r \times \frac{1}{r} = 1.$$

Q) If $x = r\sin\theta\cos\phi$ $y = r\sin\theta\cos\phi$ $z = r\cos\theta$

S.T $\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2\sin\theta.$

$J(x, y$ ... $|\sin$

$\Rightarrow -r^2\sin^3\theta\sin\phi\cos\theta + r^2\sin\theta\sin\phi\cos\phi\,(\sin^2\theta)$

$\Rightarrow -r^2\sin^3\theta\sin\phi\cos\theta + r^2\sin^3\theta\sin\phi\cos\theta$

Q) If $u = x^2 - y^2$, $v = 2xy$ where $x = r\cos\theta$, $y = r\sin\theta$ then find $\frac{\partial(u, v)}{\partial(r, \theta)} = 4r^3$.

$u = (r^2\cos^2\theta) - (r^2\sin^2\theta)$ $\quad v = 2(r^2\sin\theta\cos\theta)$

$= r^2(\cos^2\theta - \sin^2\theta)$

$\frac{\partial u}{\partial r} = \cos^2\theta - \sin^2\theta\,(2r)$

$= \cos 2\theta\,(2r)$

$\frac{\partial v}{\partial r} = 2\sin\theta\cos\theta\,(2r)$

$= 2r\sin 2\theta$.

$\frac{\partial u}{\partial \theta} = r^2\left[\frac{\partial}{\partial\theta}(\cos 2\theta)\right]$

$= r^2\left[-\sin 2\theta\,(2)\right]$

$= -2r^2\sin 2\theta$.

$\frac{\partial v}{\partial \theta} = r^2\left(\frac{\partial}{\partial\theta}(\sin 2\theta)\right)$

$= r^2(\cos 2\theta\,(2))$

$= 2r^2\cos 2\theta$

$$J\left(\frac{u, v}{r, \theta}\right) = \begin{vmatrix} 2r\cos 2\theta & -2r^2\sin 2\theta \\ 2r\sin 2\theta & 2r^2\cos 2\theta \end{vmatrix}$$

$$= 4r^3(\cos^2 2\theta) + 4r^3\sin^2 2\theta$$

$$J\left(\frac{u, v}{r, \theta}\right) = 4r^3$$

Q) If $x = e^r\sec\theta$ $y = e^r\tan\theta$ prove that $\frac{\partial(x, y)}{\partial(r, \theta)} \times \frac{\partial(r, \theta)}{\partial(x, y)} = 1$.

$\frac{\partial x}{\partial r} = \sec\theta(e^r)$ $\quad \frac{\partial x}{\partial \theta} = e^r\sec\theta\tan\theta$

$\frac{\partial y}{\partial r} = \tan\theta(e^r)$ $\quad \frac{\partial y}{\partial \theta} = e^r\sec^2\theta$.

$$J\left(\frac{x, y}{r, \theta}\right) = \begin{vmatrix} e^r\sec\theta & e^r\sec\theta\tan\theta \\ e^r\tan\theta & e^r\sec^2\theta \end{vmatrix}$$

$$= e^{2r}\sec^3\theta - e^{2r}\sec\theta\tan^2\theta.$$

$$= e^{2r}\sec\theta.$$

$$x^2 - y^2 = e^{2r}(\sec^2\theta - \tan^2\theta)$$

$$x^2 - y^2 = e^{2r}$$

$$\log(x^2-y^2) = 2r$$

$$r = \frac{\log(x^2-y^2)}{2}$$

$$\frac{\partial r}{\partial x} = \frac{1}{\cancel{2}}\frac{1}{x^2-y^2}(\cancel{2}x)$$

$$\frac{\partial r}{\partial x} = \frac{x}{x^2-y^2}$$

$$\frac{\partial r}{y} = \frac{1}{\cancel{2}}\frac{1}{x^2-y^2}(-\cancel{2}y)$$

$$= \frac{-y}{x^2-y^2}$$

$$\frac{\partial r}{\partial y} = \frac{y}{y^2-x^2}$$

$$\frac{x}{y} = \frac{\sec\theta}{\tan\theta}$$

$$\frac{x}{y} = \frac{1}{\cancel{\cos\theta}} \times \frac{\cancel{\cos\theta}}{\sin\theta}$$

$$\frac{y}{x} = \sin\theta$$

$$\theta = \sin^{-1}\left(\frac{y}{x}\right)$$

$$\frac{\partial\theta}{\partial x} = \frac{1}{\sqrt{1-\frac{y^2}{x^2}}}\left(-\frac{y}{x^2}\right)$$

$$= \frac{1}{\frac{\sqrt{x^2-y^2}}{\cancel{x}}}\left(-\frac{y}{x^{\cancel{2}}}\right)$$

$$= \frac{-y}{\sqrt{x^2-y^2}}$$

$$= \frac{-y}{x\sqrt{(e^r)^2}}$$

$$= \frac{-y}{xe^r}$$

$$\frac{\partial\theta}{\partial y} = \frac{1}{\sqrt{\frac{x^2-y^2}{x^2}}}\left(\frac{1}{x}\right)$$

$$= \frac{\cancel{x}}{e^r}\left(\frac{1}{\cancel{x}}\right)$$

$$= \frac{1}{e^r}$$

$\frac{x,y)}{r,\theta)}$

$$J\left(\frac{r,\theta}{x,y}\right) = \begin{vmatrix} \dfrac{x}{x^2-y^2} & \dfrac{y}{y^2-x^2} \\ \\ \dfrac{-y}{xe^{r}} & \dfrac{1}{e^{r}} \end{vmatrix}$$

$$= \frac{x}{e^{r}(x^2-y^2)} + \frac{y^2}{xe^{r}(y^2-x^2)}$$

$$= \frac{1}{e^{r}}\left(\frac{x^2 - y^2}{x(x^2-y)}\right)$$

$$= \frac{1}{e^{r}}\left(\frac{1}{x}\right)$$

$$= \frac{1}{e^{r}(e^{r}\sec\theta)}$$

$$= \frac{1}{e^{2r}\sec\theta}$$

$$J\left(\frac{x,y}{r,\theta}\right) J\left(\frac{r,\theta}{x,y}\right) = e^{2r}\sec\theta \times \frac{1}{e^{2r}\sec\theta} = 1$$

Q) If $x = u(1-v)$, $y = uv$ P.T $JJ' = 1$

$$J\left(\frac{x, y}{u, v}\right) \Rightarrow \frac{\partial x}{\partial u} = (1-v) \qquad \frac{\partial x}{\partial v} = \frac{\partial}{\partial v}(u - uv)$$

$$= 0 - u$$

$$= -u.$$

$$\frac{\partial y}{\partial u} = v \qquad \frac{\partial y}{\partial v} = u$$

$$\therefore J\left(\frac{x, y}{u, v}\right) = \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix} = u - \cancel{uv} + \cancel{uv}$$

$$= u$$

$$\Rightarrow v = \frac{y}{u}, \quad x = u\left(1 - \frac{y}{u}\right) \Rightarrow x = u - y \Rightarrow x + y = u$$

$$\Rightarrow v = \frac{y}{x+y}$$

$$\frac{\partial u}{\partial x} = 1, \quad \frac{\partial u}{\partial y} = 1, \quad \frac{\partial v}{\partial x} = \frac{(x+y)(0) - y(1)}{(x+y)^2} = \frac{-y}{(x+y)^2}$$

$$\frac{\partial v}{\partial y} = \frac{(x+y)1 - y(1)}{(x+y)^2} = \frac{x + y - y}{(x+y)^2} = \frac{x}{(x+y)^2}$$

$$\Rightarrow J\left(\frac{u, v}{x, y}\right) = \begin{vmatrix} 1 & 1 \\ \frac{-y}{(x+y)^2} & \frac{x}{(x+y)^2} \end{vmatrix} = \frac{x}{(x+y)^2} + \frac{y}{(x+y)^2}$$

$$= \frac{1}{x+y} = \frac{1}{u}$$

$$\therefore JJ' = u \times \frac{1}{u} = 1$$

Q) If $x = uv$, $y = \frac{u}{v}$ S.T $\frac{\partial(x,y)}{\partial(u,v)} \times \frac{\partial(u,v)}{\partial(x,y)} = 1$

Q) If $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$

Find $\frac{\partial(r,\theta,\phi)}{\partial(x,y,z)}$

Functional Dependence:

If the func^n u & v of the independent variables x & y are functionally dependent, then $\frac{\partial(u,v)}{\partial(x,y)} = 0$, then u & v are said to be functionally dependent.

If $\frac{\partial(u,v)}{\partial(x,y)} \neq 0$, then u & v are said to be funct^{n}ally independent.

$\times \frac{\partial(u,v)}{\partial(x,y)} = 1$

$, z = r\cos\theta$

the independent

pendent,

said to be

to be

Q. If $x = uv$, $y = \frac{u}{v}$ show that $\frac{\partial(x,y)}{\partial(u,v)} \frac{\partial(u,v)}{\partial(x,y)} = 1$

$\Rightarrow \frac{\partial x}{\partial u} = v \qquad \frac{\partial x}{\partial v} = u. \qquad \frac{\partial y}{\partial u} = \frac{v(1) - u(0)}{v^2} = \frac{v}{v^2} = \frac{1}{v}$

$\frac{\partial y}{\partial v} = \frac{v(0) - u(1)}{v^2} = \frac{-u}{v^2}$

$$J\left(\frac{x,y}{u,v}\right) = \begin{vmatrix} v & u \\ \frac{1}{v} & -\frac{u}{v^2} \end{vmatrix}$$

$$= \cancel{v}\left(-\frac{u}{v^{\cancel{2}}}\right) - \frac{u}{v} = -\frac{2u}{v}$$

$\Rightarrow u = vy \qquad x = uv = vy(v) \Rightarrow v^2 y = x$

$\Rightarrow v^2 = \frac{x}{y} \Rightarrow v = \sqrt{\frac{x}{y}}$

$\Rightarrow u = \sqrt{\frac{x}{\cancel{y}} \times y^{\cancel{2}}} = \sqrt{xy}.$

$\frac{\partial u}{\partial x} = \frac{1}{2\sqrt{xy}}(y) = \frac{y}{2\sqrt{xy}} \qquad \frac{\partial u}{\partial y} = \frac{1}{2\sqrt{xy}}(x) = \frac{x}{2\sqrt{xy}} = \frac{x}{2u}.$

$\frac{\partial u}{\partial x} = \frac{y}{2u}. \qquad \frac{\partial u}{\partial y} = \frac{x}{2u}.$

$\Rightarrow \frac{\partial v}{\partial x} = \frac{1}{2\sqrt{\frac{x}{y}}}\left(\frac{1}{y}\right) = \frac{1}{2yv} \qquad \frac{\partial v}{\partial y} = \frac{1}{2\sqrt{\frac{x}{y}}}\left(\frac{-x}{y^2}\right) = \frac{-x}{2vy^2}$

$$J\left(\frac{u,v}{x,y}\right) = \begin{vmatrix} \frac{y}{2u} & \frac{x}{2u} \\ \frac{1}{2vy} & \frac{-x}{2vy^2} \end{vmatrix}$$

$$= \frac{-x}{2vy^{\cancel{2}}} \times \frac{\cancel{y}}{2u} - \frac{x}{2u} \times \frac{1}{2vy}.$$

$$= \frac{-x}{4uvy} - \frac{x}{4uvy} \Rightarrow \frac{-2x}{4uvy} = \frac{-\cancel{2}v^{\cancel{2}}}{2\cancel{4}u\cancel{v}} = \frac{-v}{2u}$$

$$J\left(\frac{x,y}{u,v}\right) J\left(\frac{u,v}{x,y}\right) = \frac{-2u}{v} \times \frac{v}{2u} = 1.$$

Q) If $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$

show that $\dfrac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2\sin\theta$

$\dfrac{\partial x}{\partial r} = \sin\theta\cos\phi$ $\quad \dfrac{\partial x}{\partial \theta} = r\cos\phi\cos\theta$ $\quad \dfrac{\partial x}{\partial \phi} = r\sin\theta(-\sin\phi)$

$= -r\sin\theta\sin\phi$

$\dfrac{\partial y}{\partial r} = \sin\theta\sin\phi$ $\quad \dfrac{\partial y}{\partial \theta} = r\sin\phi\cos\theta$ $\quad \dfrac{\partial y}{\partial \phi} = r\sin\theta(+\cos\phi)$

$= r\sin\theta\cos\phi$

$\dfrac{\partial z}{\partial r} = \cos\theta$ $\quad \dfrac{\partial z}{\partial \theta} = r(-\sin\theta)$ $\quad \dfrac{\partial z}{\partial \phi} = 0$

$= -r\sin\theta$

$$J\left(\frac{x,y,z}{r,\theta,\phi}\right) = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\sin\phi\cos\theta & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$= \sin\theta\cos\phi(+r^2\sin^2\theta\cos\phi) - r\cos\theta\cos\phi(-r\sin\theta\cos\phi\cos\theta)$
$- r\sin\theta\sin\phi(-r\sin^2\theta\sin\phi - r\sin\phi\cos^2\theta)$

$= r^2\sin^3\theta\cos^2\phi + r^2\sin\theta\cos^2\phi\cos^2\theta + r^2\sin^3\theta\sin^2\phi$
$+ r^2\sin^2\phi\sin\theta\cos^2\theta.$

$$= r^2 \sin^3\theta(\cos^2\phi + \sin^2\phi) + r^2 \sin\theta \cos^2\theta(\sin^2\phi + \cos^2\phi)$$

$$= r^2 \sin^3\theta + r^2 \sin\theta \cos^2\theta$$

$$= r^2 \sin\theta(\sin^2\theta + \cos^2\theta)$$

$$J\left(\frac{x, y, z}{r, \theta, \phi}\right) = r^2 \sin\theta.$$

$$J\left(\frac{x, y, z}{r, \theta, \phi}\right) \times J\left(\frac{r, \theta, \phi}{x, y, z}\right) = 1.$$

$$\therefore J\left(\frac{r, \theta, \phi}{x, y, z}\right) = \frac{1}{J\left(\frac{x, y, z}{r, \theta, \phi}\right)}$$

$$= \frac{1}{r^2 \sin\theta}$$

$$= \frac{\operatorname{cosec}\theta}{r^2}$$

$$\therefore J\left(\frac{r, \theta, \phi}{x, y, z}\right) = \frac{\operatorname{cosec}\theta}{r^2}.$$

Q) If $u = \dfrac{x+y}{1-xy}$ and $v = \tan^{-1}x + \tan^{-1}y$

P.T u & v are F.D & find the relatⁿ b/w them.

$$\frac{\partial u}{\partial x} = \frac{(1-xy)(1) - (x+y)(-y)}{(1-xy)^2} \qquad \frac{\partial u}{\partial y} = \frac{(1-xy)(1) - (x+y)(-x)}{(1-xy)^2}$$

$$\frac{\partial u}{\partial x} = \frac{1 - \cancel{xy} + \cancel{xy} + y^2}{(1-xy)^2} \qquad = \frac{1 - \cancel{xy} + x^2 + \cancel{xy}}{(1-xy)^2}$$

$$\frac{\partial u}{\partial x} = \frac{1+y^2}{(1-xy)^2} \qquad \frac{\partial u}{\partial y} = \frac{1+x^2}{(1-xy)^2}$$

$$\frac{\partial v}{\partial x} = \frac{1}{1+x^2} \qquad \frac{\partial v}{\partial y} = \frac{1}{1+y^2}$$

$$\therefore J\left(\frac{u,v}{x,y}\right) = \begin{vmatrix} \dfrac{1+y^2}{(1-xy)^2} & \dfrac{1+x^2}{(1-xy)^2} \\ \dfrac{1}{1+x^2} & \dfrac{1}{1+y^2} \end{vmatrix}$$

$$= \frac{1}{(1-xy)^2} - \frac{1}{(1-xy)^2}$$

$$J\left(\frac{u,v}{x,y}\right) = 0 \qquad \therefore u \text{ \& } v \text{ are F.D.}$$

$$v = \tan^{-1}\left(\frac{x+y}{1-xy}\right) = \tan^{-1}u \Rightarrow u = \tan v$$

Q) P.T the

are func

relatⁿ b/

$\frac{\partial u}{\partial x} = 1$

$\frac{\partial v}{\partial x} = y+z$

$\frac{\partial w}{\partial x} = 2x$

$J\left(\frac{u,v,}{x,y,}\right.$

$= 1($

-1

$+1$

$= 2x$

$+2$

$\therefore J($

tan'y

relatⁿ b/w them.

$\frac{(1-xy)(1)-(x+y)(-x)}{(1-xy)^2}$

$= \frac{1-xy+x^2+xy}{(1-xy)^2}$

$= \frac{1+x^2}{(1-xy)^2}$

Q) P.T the func's $u = x+y+z$, $v = xy+yz+xz$

$w = x^2+y^2+z^2$

are functionally dependent & the find the relatⁿ b/w them.

$\frac{\partial u}{\partial x} = 1 \quad \frac{\partial u}{\partial y} = 1 \quad \frac{\partial u}{\partial z} = 1$

$\frac{\partial v}{\partial x} = y+z \quad \frac{\partial v}{\partial y} = x+z \quad \frac{\partial v}{\partial z} = y+x$

$\frac{\partial w}{\partial x} = 2x \quad \frac{\partial w}{\partial y} = 2y \quad \frac{\partial w}{\partial z} = 2z$

$$J\left(\frac{u,v,w}{x,y,z}\right) = \begin{vmatrix} 1 & 1 & 1 \\ y+z & x+z & x+y \\ 2x & 2y & 2z \end{vmatrix}$$

$= 1(2xz + 2z^2 - 2xy - 2y^2)$

$-1(2yz + 2z^2 - 2x^2 - 2xy)$

$+1(2y^2 + 2yz - 2x^2 - 2xz)$

$= \cancel{2xz} + \cancel{2z^2} - \cancel{2xy} - \cancel{2y^2} - 2yz - \cancel{2z^2} + \cancel{2x^2} + \cancel{2xy}$
$+ \cancel{2y^2} + \cancel{2yz} - \cancel{2x^2} - \cancel{2xz}$

$= 0$

$\therefore J\left(\frac{u,v,w}{x,y,z}\right) = 0 \quad \therefore u, v, w$ are F.D.

$$(u)^2 = w + 2v$$

$$\Rightarrow u^2 = w + 2v$$

Q) Prove that $u = \dfrac{x^2 - y^2}{x^2+y^2}$ $v = \dfrac{2xy}{x^2+y^2}$ are F.D and find the relat^n b/w them.

$$\frac{\partial u}{\partial x} = \frac{x^2+y^2(2x) - (x^2-y^2)(2x)}{(x^2+y^2)} \qquad \frac{\partial u}{\partial y} = \frac{(x^2+y^2)(-2y) - (x^2-y^2)(2y)}{(x^2+y^2)^2}$$

$$= \frac{\cancel{2x^3} + 2xy^2 - \cancel{2x^3} + 2xy^2}{(x^2+y^2)} \qquad = \frac{-2yx^2 \cancel{- 2y^3} - 2yx^2 \cancel{+ 2y^3}}{(x^2+y^2)^2}$$

$$= \frac{4xy^2}{(x^2+y^2)^2} \qquad = \frac{-4yx^2}{(x^2+y^2)^2}$$

$$\frac{\partial v}{\partial x} = \frac{x^2+y^2(2y) - 2xy(2x)}{(x^2+y^2)^2} \qquad \frac{\partial v}{\partial y} = \frac{x^2+y^2(2x) - 2xy(2y)}{(x^2+y^2)^2}$$

$$= \frac{2yx^2 + 2y^3 - 4x^2y}{(x^2+y^2)^2} \qquad = \frac{2x^3 + 2xy^2 - 4xy^2}{(x^2+y^2)^2}$$

$$= \frac{2y^3 - 2x^2y}{(x^2+y^2)^2} \qquad = \frac{2x^3 - 2xy^2}{(x^2+y^2)^2}$$

2 are F.D

$$\frac{\ldots+y^2)(-2y)-(x^2-y^2)(2y)}{(x^2+y^2)^2}$$

$$\frac{\ldots yx^2 \cancel{-2y^3} - 2yx^2 \cancel{+2y^3}}{(x^2+y^2)^2}$$

$$\frac{-4yx^2}{(x^2+y^2)^2}$$

$$\frac{\ldots(2x)-2xy(2y)}{(x^2+y^2)^2}$$

$$\frac{\ldots xy^2 - 4xy^2}{(\ldots+y^2)^2}$$

$$\frac{\ldots 2xy^2}{(\ldots y^2)^2}$$

$$J = \begin{vmatrix} \dfrac{4xy^2}{(x^2+y^2)^2} & \dfrac{-4yx^2}{(x^2+y^2)^2} \\ \dfrac{2y^3-2x^2y}{(x^2+y^2)^2} & \dfrac{2x^3-2xy^2}{(x^2+y^2)^2} \end{vmatrix}$$

$$\Rightarrow \frac{\cancel{8x^4y^2} - \cancel{8x^2y^4} + \cancel{8y^4x^2} - \cancel{8x^4y^2}}{(x^2+y^2)^4}$$

$$J\left(\frac{u,v}{x,y}\right) = 0$$

u, v are ∴ F.D

$$x^2+y^2 = \frac{x^2-y^2}{u} \Rightarrow V = \frac{2xy}{\cancel{x^2-y^2}} \times \cancel{u}$$

$$V = u\left(\frac{2xy}{x^2-y^2}\right).$$

$$u^2+v^2 = \frac{(x^2-y^2)^2}{(x^2+y^2)^2} + \frac{4x^2y^2}{(x^2+y^2)^2}$$

$$= \frac{(x^2-y^2)^2+4x^2y^2}{(x^2+y^2)^2}$$

$$= \frac{(x^2+y^2)^2}{(x^2+y^2)^2} = 1$$

Q) P.T $u = 2x - y + 3z$ $v = 2x - y - z$ $w = 2x - y + z$

are F.D and find functional relation.

$$\frac{\partial u}{\partial x} = 2 \qquad \frac{\partial u}{\partial y} = -1 \qquad \frac{\partial u}{\partial z} = 3$$

$$\frac{\partial v}{\partial x} = 2 \qquad \frac{\partial v}{\partial y} = -1 \qquad \frac{\partial v}{\partial z} = -1$$

$$\frac{\partial w}{\partial x} = 2 \qquad \frac{\partial w}{\partial y} = -1 \qquad \frac{\partial w}{\partial z} = 1$$

$$J\left(\frac{u, v, w}{x, y, z}\right) = \begin{vmatrix} 2 & -1 & 3 \\ 2 & -1 & -1 \\ 2 & -1 & 1 \end{vmatrix}$$

$$= 2(-1-1) + 1(2+2) + 3(-2+2)$$

$$= -4 + 4 = 0$$

$$\therefore J\left(\frac{u, v, w}{x, y, z}\right) = 0 \quad \therefore u, v, w \text{ are F.D}$$

$$u + v - 2w = 2x - y + 3z + 2x - y - z - 4x + 2y - 2z$$

$$= \cancel{4x} - \cancel{2y} + \cancel{2z} - \cancel{4x} + \cancel{2y} - \cancel{2z}$$

$$= 0$$

$$\therefore u + v - 2w = 0$$

$$\Rightarrow u + v = 2w$$

Maximum and Minimum of Func^{n}s of two Variables.

→ Let $f(x,y)$ be a func^n of two variables x & y, at $x=a$, $y=b$, $f(x,y)$ is said to have maximum or minimum value if $f(a,b) > f(a+h, b+k)$ or $f(a,b) < f(a+h, b+k)$ respectively, where h & k are small values.

→ Extreme value: $f(a,b)$ is said to be an extreme value of f if it is a maximum or minimum value.

<u>Working rule:</u>

1) Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ and equate each to zero & solve these two eq^n we get the pair of points $(a_1,b_1), (a_2,b_2), \ldots$ these are called <u>stationary</u> points.

2) $r = \frac{\partial^2 f}{\partial x^2}$ $s = \frac{\partial^2 f}{\partial x \partial y}$ $t = \frac{\partial^2 f}{\partial y^2}$

3) i) $rt - s^2 > 0$ and $r < 0$ at (a_1, b_1) then (a_1, b_1) is a point of maximum and $f(a_1, b_1)$ is a maximum value.

ii) $rt - s^2 > 0$ and $r > 0$ at (a_1, b_1) then (a_1, b_1) is a point of minimum and $f(a_1, b_1)$ is a minimum value.

iii) $rt - s^2 < 0$ at (a_1, b_1) then neither a maximum nor a minimum at a_1, b_1 exists. In this case the point (a_1, b_1) is a saddle point.

iv) $rt - s^2 = 0$ at (a_1, b_1), no conclusion can be drawn about maximum or minimum ~~an~~

↳ Similarly find the other pair of values $(a_2, b_2), (a_3, b_3), \ldots\ldots$

Q) Find the maximum and minimum values of

$$f(x,y) = x^3 + 3xy^2 - 3x^2 - 3y^2 + 4.$$

$$\frac{\partial f}{\partial x} = 3x^2 + 3y^2 - 6x$$

$$\frac{\partial f}{\partial y} = 3x(2y) - 6y = 6xy - 6y.$$

$$\Rightarrow 3x^2 + 3y^2 - 6x = 0$$

$$6xy - 6y = 0.$$

$$y(6x-6) = 0.$$

$$(y-0)(6x-6) = 0$$

$$y = 0, \quad x = 1$$

i) $3x^2 - 6x = 0$

$$x(3x-6) = 0$$

$$x = 0, \quad x = 2.$$

ii) $3 + 3y^2 - 6 = 0.$

$$3y^2 - 3 = 0$$

$$y^2 - 1 = 0$$

$$y^2 = 1$$

$$y = 1, \quad y = -1$$

$\Rightarrow$ Stationary points are, $(0,0), (2,0), (1,1), (1,-1)$

ues of

$r = 6x-6 \qquad s = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) \qquad t = 6x-6.$

$s = 6y$

$s^2 = 36y^2$

$\therefore \cancel{rt-s^2} \Rightarrow \cancel{6x-6}$

$\Rightarrow rt - s^2 \Rightarrow (6x-6)^2 - (36y^2)$

$\Rightarrow 36x^2 + 36 - 72x - 36y^2.$

$\Rightarrow 36x^2 - 36y^2 - 72x + 36.$

$\therefore$ i) at $(0,0)$

$\Rightarrow (6x-6)^2 - (36y^2) \Rightarrow 36 > 0$ & $6(0)-6 = -6$

$\therefore r < 0.$

$\therefore (0,0)$ is a maximum pt.

$\therefore$ maximum value is $f(0,0) = 4.$

ii) at $(2,0)$

$\Rightarrow (12-6)^2 - 36(0) = 36 > 0$ & $6(2)-6 \Rightarrow 6 > 0.$

$\therefore (2,0)$ is a minimum pt

$\therefore$ minimum value is $f(2,0) = 8 + 4 - 12$

$= 0.$

1)

iii) at $(1,1)$

$(6-6)^2 - 36(1) = -36 < 0$

at $(1,1)$ $f(x,y)$ has neither max nor min.

iv) at $(1,-1)$

$\Rightarrow (6(1)-6)^2 - 36(-1)^2 = -36 < 0.$

$\therefore$ at $(1,-1)$ ~~no~~ $f(x,y)$ has neither max nor min.

Q) Discuss the maxima and minima of

$x^2y + xy^2 - axy.$

$f(x,y) = x^2y + xy^2 - axy.$

$\frac{\partial f}{\partial x} = 2xy + y^2 - ay \qquad \frac{\partial f}{\partial y} = x^2 + 2xy - ax$

$$\begin{array}{rl} & 2xy + y^2 - ay = 0 \\ (-) & 2xy + x^2 - ax = 0 \\ \hline & y^2 - x^2 - ay + ax = 0 \end{array}$$

$\Rightarrow x^2 - y^2 + ay - ax = 0.$

$\cancel{x(x-a) + y(a-y) = 0}$

$\cancel{(x-b)(x-a) + (y-b)(a-y) = 0}$

$(y^2 - x^2) - a(y - x) = 0$

$(y-x)(y+x) - a(y-x) = 0$

$(y-x)(y+x-a) = 0$

$\therefore y - x = 0 \quad , \quad y + x - a = 0$

$y = x \quad , \quad y + x = a.$

i) $2xy + y^2 - ay = 0$

$2x^2 + x^2 - ax = 0$

$3x^2 - ax = 0$

$x(3x - a) = 0$

$x = 0, \; x = \frac{a}{3}$

$\therefore$ stationary points are $(0,0), \left(\frac{a}{3}, \frac{a}{3}\right)$

ii) $2xy + y^2 - ay = 0$

$2x(a-x) + (a-x)^2 - ay = 0$

$\cancel{2ax} - 2x^2 + a^2 + x^2 - \cancel{2ax} - ay = 0.$

$-x^2 + a^2 - ay = 0$

$x^2 - a^2 + ay = 0$

$x^2 - a^2 + a(a-x) = 0$

$x^2 - \cancel{a^2} + \cancel{a^2} - ax = 0$

$x^2 - ax = 0$

$x(x-a) = 0$

$x = 0, \; x = a$

$\therefore$ Stationary pts are $(0, a), (a, 0)$

$r = 2y \qquad s = 2x + 2y - a \qquad t = 2x.$

$s^2 = (2x + 2y - a)^2$

i) at pt $(0,0)$

$\Rightarrow rt - s^2 = 4(0)(0) - (2(0) + 2(0) - a)^2$

$= -a^2 < 0.$

$\therefore$ at $(0,0)$ ~~neithe~~ $f(x,y)$ has neither max nor min.

(ii) at $\left(\frac{a}{3}, \frac{a}{3}\right)$

$$rt - s^2 = 4\left(\frac{a}{3}\right)\left(\frac{a}{3}\right) - \left(2\left(\frac{a}{3}\right) + 2\left(\frac{a}{3}\right) - a\right)^2$$

$$= \frac{4a^2}{9} - \left(\frac{4a - 3a}{3}\right)^2$$

$$= \frac{4a^2}{9} - \frac{a^2}{9} = \frac{3a^2}{9} = \frac{a^2}{3} > 0$$

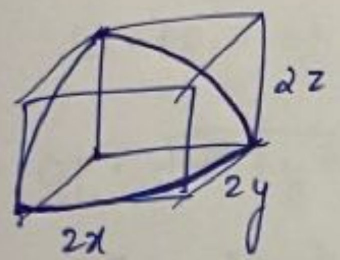
$$r = 2\left(\frac{a}{3}\right) = \frac{2a}{3}$$

If $a > 0$ then $r > 0$ $\therefore$ minimum

If $a < 0$ then $r < 0$ $\therefore$ maximum

Q1) Find the value of the largest parallelopiped that can be inscribed in the ellipsoid,

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$



length $= 2x$

breadth $= 2y$

height $= 2z$

$$f(x,y,z) = V = (2x)(2y)(2z) = 8xyz$$

$$\phi(x,y,z) = \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1$$

$$F(x,y,z) = 8xyz + \lambda\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)$$

$$F_x = 8yz + \frac{\lambda}{a^2}(2x) = 0$$

$$F_y = 8xz + \frac{\lambda}{b^2}(2y) = 0$$

$$F_z = 8xy + \frac{\lambda}{c^2}(2z) = 0.$$

$$\therefore \lambda = \frac{-8yz(a^2)}{2x}, \quad \lambda = -\frac{8xz(b^2)}{2y}, \quad \lambda = \frac{-8xy(c^2)}{2z}$$

$$\therefore \frac{-\cancel{8}y\cancel{z}(a^2)}{\cancel{2}x} = \frac{-\cancel{8}x\cancel{z}(b^2)}{\cancel{2}y} = \cancel{-\frac{8xy(c^2)}{2z}}$$

$$\Rightarrow y^2a^2 = x^2b^2 \qquad \Rightarrow z^2a^2 = x^2c^2.$$

$$\Rightarrow z^2b^2 = y^2c^2$$

$\therefore \quad y^2a^2 = x^2b^2 \rightarrow (1) \qquad z^2a^2 = x^2c^2$

$z^2b^2 = y^2c^2 \qquad \dfrac{x^2}{a^2} = \dfrac{z^2}{c^2}$

$\Rightarrow \dfrac{y^2}{b^2} = \dfrac{z^2}{c^2}.$

$\Rightarrow$ From (1) $\dfrac{x^2}{a^2} = \dfrac{y^2}{b^2}$

$\therefore$

$$\therefore \quad \frac{x^2}{a^2} = \frac{y^2}{b^2} = \frac{z^2}{c^2}.$$

$$\therefore \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

$$\Rightarrow \frac{x^2}{a^2} + \frac{x^2}{a^2} + \frac{x^2}{a^2} = 1$$

$$\Rightarrow \frac{3x^2}{a^2} = 1.$$

$$\therefore 3x^2 = a^2$$

$$x = \pm \frac{a}{\sqrt{3}}$$

$$y = \pm \frac{b}{\sqrt{3}}$$

$$z = \pm \frac{c}{\sqrt{3}}$$

$$\therefore (x, y, z) = \left(\frac{a}{\sqrt{3}}, \frac{b}{\sqrt{3}}, \frac{c}{\sqrt{3}} \right)$$

max value $= V = 8xyz = 8\left(\dfrac{a}{\sqrt{3}}\right)\left(\dfrac{b}{\sqrt{3}}\right)\left(\dfrac{c}{\sqrt{3}}\right)$

$$= \frac{8abc}{3\sqrt{3}}.$$

Q) Find the pt on the plane $x+2y+3z=4$ that is closed to the origin.

Let $p(x,y,z)$ be any point on the plane.

$O = (0,0,0)$

$OP = \sqrt{x^2+y^2+z^2}$

$OP^2 = x^2+y^2+z^2$

$f(x,y,z) = x^2+y^2+z^2$

$\phi(x,y,z) = x+2y+3z=4$

$F(x,y,z) = x^2+y^2+z^2+\lambda x+\lambda 2y+3\lambda z.$

$F_x = 2x+\lambda$

$F_y = 2y+2\lambda$

$F_z = 2z+3\lambda$

$\therefore \lambda = -2x, \quad \lambda = \frac{-2y}{2}, \quad \cancel{2}\lambda = \frac{-2z}{3}$

$\lambda = -y \quad \lambda = -\frac{2z}{3}$

$-2x = -y \quad -y = -\frac{2z}{3} \quad -\cancel{2}x = -\frac{\cancel{2}z}{3}$

$2x = y \quad 3y = 2z \quad 3x = z.$

$x = \frac{y}{2} \quad \frac{y}{2} = \frac{z}{3} \quad \frac{z}{3} = x$

$\therefore x = \frac{y}{2} = \frac{z}{3} \Rightarrow 6x = 3y = 2z$

$\therefore \quad x + 2y + 3z = 4$

$\Rightarrow \quad x + 2(2x) + 3(3x) = 4$

$\Rightarrow \quad x + 4x + 9x = 4$

$$14x = 4$$

$$x = \frac{\cancel{4}^{\,2}}{\cancel{14}_{\,7}} = \frac{2}{7}$$

$$y = 2x = 2\left(\frac{4}{14}\right) = \frac{4}{7}$$

$$z = 3\left(\frac{4}{14}\right) = \frac{\cancel{12}\,6}{\cancel{14}\,7} = \frac{6}{7}$$

$$\therefore \quad f(x, y, z) = x^2 + y^2 + z^2$$

$$= \frac{4}{49} + \frac{16}{49} + \frac{36}{49}$$

$$f(x, y, z) = \frac{\cancel{56}\,8}{\cancel{49}\,7} = \frac{8}{7}$$

$$\Rightarrow (x, y, z) = \left(\frac{2}{7}, \frac{4}{7}, \frac{6}{7}\right) = pt$$

1Q) Find the max & min distances of the pt $(3, 4, 12)$ from the sphere $x^2 + y^2 + z^2 = 1$.

2Q) Find the pt on the plane $3x + 2y + z = 12$ which is nearest to the origin.

3Q) Find the max value of $u = x^2 y^3 z^4$ if $2x + 3y + 4z = a$

48) Find the shortest distance from the origin to the surface $xyz^2=2$

58) Find the pts on the surface $z^2 = xy+1$ that are nearest to the origin.

68) Find three +ve numbers whose sum is 100 and whose product is max.

$$\phi(x,y,z) = x+y+z = 100$$

$$f(x,y,z) = xyz$$

$$F(x,y,z) = xyz + (x+y+z)\lambda$$

$$F_x = yz+\lambda \quad F_y = xz+\lambda \quad F_y = xy+\lambda$$

$$\lambda = -yz, \ \lambda = -xz, \ \lambda = -xy.$$

$$y = x, \ z = y, \ x = z$$

$$\therefore \ x = y = z.$$

$$\therefore \ x+x+x = 100$$

$$3x = 100$$

$$\Rightarrow x = \frac{100}{3}, y = \frac{100}{3}, z = \frac{100}{3}.$$

$$f(x,y,z) = xyz = \left(\frac{100}{3}\right)^3 = \text{max value}.$$

the pt

$^2 = 1.$

$z = 12$

if

7Q) Find the maxima and minima of

$$xy + \frac{a^3}{x} + \frac{a^3}{y}.$$

8Q) Find the max and min values of $f(x,y)$

$$= x^3 y^2 (1-x-y).$$

9Q) A Rectangular box open at the top is to have volume 32 cubic feets. Find the dimension of the box required least material for its construct^n.

$$f(x,y,z) = 2xy + 2zx + yz$$

$$\phi(x,y,z) = xyz = 32.$$

$$\Rightarrow F(x,y,z) = 2xy + 2zx + yz + \lambda(xyz)$$

$$F_x = 2y + 2z + \lambda yz = 0$$

$$F_y = 2x + z + \lambda xz = 0$$

$$F_z = 2x + y + \lambda xy = 0$$

$$\therefore \lambda = \frac{-2y-2z}{yz}, \lambda = \frac{-2x-z}{xz}, \lambda = \frac{-2x-y}{xy}.$$

$$\frac{-2y-2z}{yz} = \frac{-2x-z}{xz}$$

$$\frac{-2x-z}{xz} = \frac{-2x-y}{xy}$$

$-2xy-2zx = -2x...$

$$-\cancel{2x} = -z$$
$$2x = z.$$

$$\Rightarrow \frac{-2y-2z}{y\cancel{z}} = \frac{-2x-z}{x\cancel{z}}, \quad \frac{-2x-z}{\cancel{x}z} = \frac{-2x-y}{\cancel{x}y}$$

$$\Rightarrow -2y\cancel{x} - 2xz = -\cancel{2x}y - yz \qquad \Rightarrow -2xy - zy = -2xz - yz$$

$$\Rightarrow -2x\cancel{z} = -y\cancel{z} \qquad -\cancel{2x}y = -\cancel{2x}z$$

$$\Rightarrow 2x = y \qquad y = z$$

$$\therefore 2x = y = z$$

$$\phi(x, y, z) = xyz = 32$$

$$\Rightarrow x(2x)(2x) = 32$$

$$4x^3 = 32 \; 8$$

$$x = 2., \; y = 4, \; z = 4$$

$\therefore$ Dimensions of the box are $(x = 2, y = 4, z = 4)$

Q1) $\phi(x,y,z) = x^2+y^2+z^2 = 1$

$P = (3, 4, 12)$

$$QP = \sqrt{(x-3)^2+(y-4)^2+(z-12)^2}$$

$$= \sqrt{x^2+9-6x+y^2+16-8y+z^2+144-24z}$$

$$= \sqrt{x^2+y^2+z^2-6x-8y-24z+169}$$

$$\therefore QP^2 = x^2+y^2+z^2-6x-8y-24z+169.$$

$$\therefore f(x,y,z) = x^2+y^2+z^2-6x-8y-24z+169$$

$$F(x,y,z) = x^2+y^2+z^2-6x-8y-24z+169 + \lambda(x^2+y^2+z^2)$$

$$F_x = 2x-6+2\lambda x = 0.$$

$$F_y = 2y-8+2y\lambda = 0$$

$$F_z = 2z-24+2\lambda z = 0$$

$$\therefore \lambda = \frac{-2x+6}{2x}, \quad \lambda = \frac{-2y+8}{2y}, \quad \lambda = \frac{-2z+24}{2z}$$

$$\frac{-2x+6}{2x} = \frac{-2y+8}{2y}$$

$$-\cancel{4xy}+12y = -\cancel{4xy}+16x.$$

$$\cancel{12}^{3}y = \cancel{16}^{2}x$$

$$3y = 2x.$$

$$\frac{y}{2} = \frac{x}{3}$$

$$\frac{-2y+8}{\cancel{2}y} = \frac{-2z+24}{\cancel{2}z}$$

$$-\cancel{2yz}+8z = -\cancel{2yz}+24y$$

$$\cancel{8}^{1}z = \cancel{24}^{3}y$$

$$z = 3y.$$

$$\frac{z}{3} = y$$

$$\Rightarrow \frac{z}{\ } = \frac{y}{\ }$$

$\therefore \frac{x}{3} = \frac{y}{2} = \frac{z}{6}.$

$2x = 3y = z$

$\therefore$

$\Rightarrow \phi(x,y,z) = x^2+y^2+z^2 = 1$

$\Rightarrow x^2 + \left(\frac{2x}{3}\right)^2 + (2x)^2 = 1$

$x^2 + \frac{4x^2}{9} + 4x^2 = 1$

$9x^2 + 4x^2 + 36x^2 = 9$

$\therefore 49x^2 = 9$

$x^2 = \frac{9}{49} \Rightarrow x = \pm\frac{3}{7}$

$\therefore y = \frac{2}{\cancel{3}}\left(\frac{\cancel{3}}{7}\right) = \pm\frac{2}{7} \ , \ z = 2\left(\frac{3}{7}\right) = \pm\frac{6}{7}.$

$\therefore Q = \left(\frac{3}{7}, \frac{2}{7}, \frac{6}{7}\right)$ or $Q = \left(-\frac{3}{7}, -\frac{2}{7}, -\frac{6}{7}\right)$

$PQ^2 = x^2+y^2+z^2-6x-8y-24z+169$

$PQ^2 = \frac{9}{49} + \frac{4}{49} + \frac{36}{49} - \frac{18}{7} - \frac{16}{7} - \frac{144}{7} + 169$

$PQ = \sqrt{\frac{1012}{7}} \Rightarrow$ minimum ~~maximum~~ distance.

$PQ^2 = \frac{9}{49} + \frac{4}{49} + \frac{36}{49} + \frac{18}{7} + \frac{16}{7} + \frac{144}{7} + 169$

$PQ = \sqrt{\frac{1368}{7}} \Rightarrow$ maximum distance

Q2) let $P = (x, y, z)$

$$OP = \sqrt{(x-0)^2+(y-0)^2+(z-0)^2}$$
$$= \sqrt{x^2+y^2+z^2}$$

$$f(x,y,z) = OP^2 = x^2+y^2+z^2$$

$$\phi(x,y,z) = 3x+2y+z = 12$$

$$\therefore F(x,y,z) = x^2+y^2+z^2+\lambda(3x+2y+z)$$

$$F_x \Rightarrow 2x+3\lambda = 0$$
$$F_y = 2y+2\lambda = 0$$
$$F_z = 2z+\lambda = 0$$

$$\therefore \lambda = -\frac{2x}{3}, \quad \lambda = -\frac{\cancel{2}y}{\cancel{2}} = -y, \quad \lambda = -2z$$

$$-\frac{2x}{3} = -y \qquad -y = -2z \qquad -\frac{\cancel{2}x}{3} = -\cancel{2}z$$

$$-2x = -3y \qquad \Rightarrow y = 2z \qquad x = 3z$$

$$\Rightarrow 2x = 3y \qquad \Rightarrow \cancel{3y} = \cancel{3 \cdot 2z} \Rightarrow 3y = 6z \qquad \frac{x}{3} = \frac{z}{2}$$

$$\Rightarrow \frac{x}{3} = \frac{y}{2} \quad \therefore 2x = 3y = 6z$$

$$\therefore \phi(x,y,z) = 3x+2y+z = 12$$

$$3x + 2\left(\frac{2x}{3}\right) + 1\left(\frac{x}{3}\right) = 12$$

$$3x + \frac{4x}{3} + \frac{x}{3} = 12$$

$$9x + 4x + x = 36$$

$$14x = 36$$

$$x = \frac{36}{14}$$

$$y = \frac{2}{3}\left(\frac{36}{14}\right) = \frac{24}{14}, \quad z = \frac{1}{3}\left(\frac{36}{14}\right)$$

$$\therefore P = \left(\frac{36}{14}, \frac{24}{14}, \frac{12}{14}\right)$$

$$\therefore P = \left(\frac{18}{7}, \frac{12}{7}, \frac{6}{7}\right)$$

Q3) $f(x, y, \ldots$

$F(x, y, z) = x \ldots$

$F_x = 2xy^3 \ldots$

$F_y = 3y^2x \ldots$

$F_z = 4z^3x \ldots$

$\therefore \lambda = -\frac{\cancel{2}xy \ldots}{\cancel{2}}$

$\Rightarrow -xy^3z^4 = - \ldots$

$y = x$

$\therefore \ldots$

$\phi(x,y,z) = \ldots$

$\therefore$ Max valu...

Q3) $f(x,y,z) = x^2y^3z^4$ $\quad \phi(x,y,z) = 2x+3y+4z = a$

$F(x,y,z) = x^2y^3z^4 + \lambda(2x+3y+4z)$

$F_x = 2xy^3z^4 + 2\lambda$

$F_y = 3y^2x^2z^4 + 3\lambda$

$F_z = 4z^3x^2y^3 + 4\lambda$

$\therefore \lambda = -\frac{\cancel{2}xy^3z^4}{\cancel{2}}, \quad \lambda = -\frac{\cancel{3}y^2x^2z^4}{\cancel{3}}, \quad \lambda = -\frac{\cancel{4}z^3x^2y^3}{\cancel{4}}$

$\Rightarrow -\cancel{x}y^{\cancel{3}}\cancel{z^4} = -\cancel{y^2}x^{\cancel{2}}\cancel{z^4}$ $\quad , \quad \cancel{y^2}\cancel{x^2}z^{\cancel{4}} = \cancel{z^3}\cancel{x^2}y^{\cancel{3}}$

$y = x$ $\quad\quad z = y$

$\therefore x = y = z$

$\phi(x,y,z) = 2x+3y+4z = a$

$2x+3x+4x = a$

$9x = a$

$x = \frac{a}{9}, \; y = \frac{a}{9}, \; z = \frac{a}{9}$

$\therefore$ Max value of $x^2y^3z^4$ is $\left(\frac{a}{9}\right)^2\left(\frac{a}{9}\right)^3\left(\frac{a}{9}\right)^4$

$\Rightarrow \left(\frac{a}{9}\right)^9$

Q4) $\phi(x,y,z) = xyz^2 = 2$

let $P = (x,y,z)$

$OP^2 = x^2+y^2+z^2$

$\therefore f(x,y,z) = x^2+y^2+z^2$

$F(x,y,z) = x^2+y^2+z^2+\lambda(xyz^2)$

$F_x = 2x + \lambda yz^2 = 0$

$F_y = 2y + \lambda xz^2 = 0$

$F_z = 2z + 2xy\lambda z = 0$

$\therefore \lambda = \frac{-2x}{yz^2}, \lambda = \frac{-2y}{xz^2}, \lambda = -\frac{\cancel{2}z}{\cancel{2}xyz}$

$\therefore \Rightarrow -\frac{\cancel{2}x}{yz^2} = -\frac{\cancel{2}y}{xz^2}$ $\qquad$ $-\frac{2y}{xz^2} = -\frac{z}{xyz}$

$\Rightarrow x^2 = y^2$ $\qquad$ $2y^2 = z^2$

$y^2 = \frac{z^2}{2}$

$\therefore x^2 = y^2 = \frac{z^2}{2}$

$\Rightarrow \phi(x,y,z) = xyz^2 = 2$

$\Rightarrow x^2y^2z^4 = 4$

$x^2(x^2)(2x^2)^2 = 4$

$x^4(4x^4) = 4$

$4x^8 = 4$

$x^8 = 1$

$x = \pm 1$

$\therefore (x,y,z) = (1,1,\frac{1}{2})$ or $(-1,-1,-\frac{1}{2})$

$\therefore OP^2 = 1+1+\frac{1}{4} = \frac{4+4+1}{4} = \frac{9}{4}$

$\therefore OP = \frac{3}{2}$ $\therefore$ Shortest distance $= \frac{3}{2}$.

Q5) $\phi(x,y,$

$P = (x,y,$

$\therefore OP^2 = x^2+y^2$

$\therefore F(x,y,z) =$

$F_x = 2x -$

$F_y = 2y$

$F_z = 2z +$

$\Rightarrow \lambda = \frac{-2x}{-y}$

$\frac{-2x}{-y} = \frac{-2y}{-x}$

$\Rightarrow \cancel{2}x^2 = \cancel{2}y^2$

$\Rightarrow x^2 = y^2$

Q7) $f(x,y) = xy + \frac{a^3}{x} + \frac{a^3}{y}$

$\frac{\partial f}{\partial x} = y + a^3\left(-\frac{1}{x^2}\right)$

$\frac{\partial f}{\partial x} = y - \frac{a^3}{x^2}$

$\frac{\partial f}{\partial y} = x + a^3\left(-\frac{1}{y^2}\right)$

$\frac{\partial f}{\partial y} = x - \frac{a^3}{y^2}$

$\therefore y - \frac{a^3}{x^2} = 0$

$y = \frac{a^3}{x^2}$

$x^2 y = a^3$

$\therefore x^{\cancel{2}} \cancel{y} = \cancel{x} y^{\cancel{2}}$

$\boxed{x = y}$

$\Rightarrow x^2(x) = a^3$

$x^3 = a^3$

$x = a, \; y = a.$

$\Rightarrow x = \frac{a^3}{y^2}$

$xy^2 = a^3.$

~~i) $ay^2 = a^3.$~~

~~$\Rightarrow \boxed{y^2 = a^2} \Rightarrow y = \pm a$~~

~~ii) $x(a^2) = a^3$~~

~~$\boxed{x = a}$~~

$\therefore$ Stationary point is (a,a), ~~$(a,-a)$~~.

$r = \frac{\partial^2 f}{\partial x^2} = -a^3\left(-\frac{2}{x^3}\right)$

$r = \frac{2a^3}{x^3}.$

$t = \frac{\partial^2 f}{\partial y^2} = -a^3\left(-\frac{2}{y^3}\right)$

$t = \frac{2a^3}{y^3}$

$s = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(x - \frac{a^3}{y^2}\right) = 1 \quad \therefore s^2 = 1.$

$rt - s^2 \Rightarrow \frac{2a^3}{x^3} \times \frac{2a^3}{y^3} - 1 \Rightarrow \frac{4a^6}{x^3y^3} - 1$

i) at (a,a)

$rt - s^2 \Rightarrow \frac{4\cancel{a^6}}{\cancel{a^3}\cancel{a^3}} - 1 \Rightarrow 3 > 0$

$r = \frac{2a^3}{x^3} = \frac{2\cancel{a^3}}{\cancel{a^3}} = 2 \Rightarrow r > 0$

$\therefore f(x,y)$ has mininum as pt (a,a)

at point $(a,-a)$.

$\Rightarrow f(a,a) = a^2 + \frac{a^{\cancel{3}^2}}{\cancel{a}} + \frac{a^{\cancel{3}^2}}{\cancel{a}} \Rightarrow 3a^2 =$ minima.

Q8) $f(x,y) = x^3y^2(1-x-y) = x^3y^2 - x^4y^2 - x^3y^3$

$\frac{\partial f}{\partial x} = 3x^2y^2 - 4x^3y^2 - 3x^2y^3$

$\frac{\partial f}{\partial y} = 2yx^3 - 2yx^4 - 3y^2x^3$

$\therefore \frac{\partial f}{\partial x} = 0$ & $\frac{\partial f}{\partial y} = 0$

$\Rightarrow x^2y^2(3-4x-3y) = 0.$

$\Rightarrow 3-4x-3y = 0. \rightarrow (1)$

$\Rightarrow xy(2x^2 - 2x^3 - 3yx^2) = 0$

$x^2(2-2x-3y) = 0$

$2-2x-3y = 0 \rightarrow (2)$

$\Rightarrow \quad 2-2x-\cancel{3y} = 0$

$(-) \quad 3-4x-\cancel{3y} = 0$

$-1+2x = 0$

$2x = 1$

$x = \frac{1}{2}.$

$2 - \cancel{2}\left(\frac{1}{\cancel{2}}\right) - 3(y) = 0$

$2-1-3y = 0$

$3y = 1$

$y = \frac{1}{3}$

$x = \frac{\partial}{\partial x}$

$= 6x$

$s =$

at P

$rt - s^2$

$\Rightarrow ($

$\Rightarrow$

$r =$

nor minimum.

minima.

$x^4y^2 - x^3y^3$

$-3(y) = 0$

$= 0$

1

$\cancel{3 - 4\left(\frac{1}{2}\right) - 3y = 0}$

$\Rightarrow$ stationary point is $\left(\frac{1}{2}, \frac{1}{3}\right)$

$\therefore$ $\cancel{r = -4}$ $\quad$ $\cancel{t = -3}$ $\quad$ $\cancel{s = \frac{\partial}{\partial x}(2 - 2x - 3y) = -2}$

$\cancel{s^2 = 4}$

$r = \frac{\partial}{\partial x}(3x^2y^2 - 4x^3y^2 - 3x^2y^3)$ $\quad$ $t = \frac{\partial}{\partial y}(2yx^3 - 2yx^4 - 3x^3y^2)$

$= 6xy^2 - 12x^2y^2 - 6xy^3$ $\quad$ $= 2x^3 - 2x^4 - 6x^3y$

$s = \frac{\partial}{\partial x}(2yx^3 - 2yx^4 - 3x^3y^2)$

$= 6yx^2 - 8yx^3 - 9x^2y^2$

$\therefore$ at pt $\left(\frac{1}{2}, \frac{1}{3}\right)$

$$rt - s^2 \Rightarrow \left(6\left(\frac{1}{2}\right)\left(\frac{1}{9}\right) - 12\left(\frac{1}{4}\right)\left(\frac{1}{9}\right) - 6\left(\frac{1}{2}\right)\left(\frac{1}{27}\right)\right)\left(2\left(\frac{1}{8}\right) - 2\left(\frac{1}{16}\right) - 6\left(\frac{1}{8}\right)\left(\frac{1}{3}\right)\right) - \left(6\left(\frac{1}{3}\right)\left(\frac{1}{4}\right) - 8\left(\frac{1}{3}\right)\left(\frac{1}{8}\right) - 9\left(\frac{1}{4}\right)\left(\frac{1}{9}\right)\right)^2$$

$$\Rightarrow \left(\frac{1}{3} - \frac{1}{3} - \frac{1}{9}\right)\left(\frac{1}{4} - \frac{1}{8} - \frac{1}{4}\right) - \left(\frac{1}{2} - \frac{1}{3} - \frac{1}{4}\right)^2$$

$$\Rightarrow \left(\frac{1}{72}\right) - \left(\frac{12 - 8 - 6}{24}\right)^2$$

$$\Rightarrow \left(\frac{1}{72}\right) - \left(-\frac{1}{12}\right)^2 \Rightarrow \frac{1}{72} - \left(\frac{1}{144}\right) \Rightarrow \frac{1}{144} > 0.$$

$\therefore r = -\frac{1}{9} < 0$ $\therefore$

$\therefore$ $f(x,y)$ has maxima at pt $\left(\frac{1}{2}, \frac{1}{3}\right)$

$$f\left(\frac{1}{2}, \frac{1}{3}\right) = \left(\frac{1}{2}\right)^3 \left(\frac{1}{3}\right)^2 \left(1 - \frac{1}{2} - \frac{1}{3}\right)$$

$$= \left(\frac{1}{8}\right)\left(\frac{1}{9}\right)\left(\frac{6-3-2}{6}\right)$$

$$= \frac{1}{8} \times \frac{1}{9} \times \frac{1}{6}.$$

$\Rightarrow$ $\frac{1}{432}$. is the maxima.